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Global Existence in L^1 for the Generalized Enskog Equation

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Various existence theorems are given for the generalized Enskog equation in R^3 and in a bounded spatial domain with periodic boundary conditions. A very general form of the geometric factor Y is allowed, including an explicit space, velocity, and time dependence. The method is based on the existence of a Liapunov functional, an analog of the H-function in the Boltzmann equation, and utilizes a weak compactness argument in L^1.

KEY WORDS: Existence theorems; nonlinear evolution equations; Enskog equation; Boltzmann equation; kinetic theory.

1. INTRODUCTION

In this paper I prove various existence theorems for the generalized Enskog equation in three dimensions. Until now, only one-dimensional and near-vacuum global results have been known, or results with unphysical simplifications of the scattering operator. The main results of this paper are contained in Theorems 2.1–2.3. I solve the Cauchy problem for the general Enskog operator by extending arguments first introduced by DiPerna and Lions.[1] Due to the many forms of the Enskog equation that are known in the literature, I provide below a brief summary of the subject.

The Enskog equation, proposed in 1921 by Enskog[2] to take account explicitly of the finite diameter of molecules, is a successful kinetic model of a dense gas consisting of hard spheres. The revised Enskog equation can be derived from the BBGKY hierarchy by computing the reduced

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N-particle distribution function from a special grand canonical formalism, and by imposing a suitable closure relation for the two-particle distribution function (see Van Beijeren and Ernst,[3] Resibois,[4] and Karkheck and Stell[5] for a recent derivation using the maximum-entropy approach). The result is

$$\frac{\partial f}{\partial f}+v\frac{\partial f}{\partial x}=E(f) \tag{1.1}$$

where $f(t, x, v)$ is the one-particle distribution function with $t \geqslant 0$, $x, v \in R^3$, and the collision operator $E(f)$ is defined by $E(f)=E^+(f)-E^-(f)$ with

$$E^+(f)=\iint_{R^3\times S^2_+} Y(x, x-a\varepsilon, n) \times f(t, x, v')\, f(t, x-a\varepsilon, w')\langle \varepsilon, v-w\rangle\, d\varepsilon\, dw \tag{1.2a}$$

$$E^-(f)=\iint_{R^3\times S^2_+} Y(x, x+a\varepsilon, n) \times f(t, x, v)\, f(t, x+a\varepsilon, w)\langle \varepsilon, v-w\rangle\, d\varepsilon\, dw \tag{1.2b}$$

Here, a denotes the hard-sphere diameter $\langle \cdot, \cdot\rangle$ is the inner product in R^3, $\varepsilon \in S^2_+ = \{\varepsilon \in R^3\colon |\varepsilon| = 1,\ \langle v-w, \varepsilon\rangle \geqslant 0\}$, and the velocities after the collision v' w' are given by

$$v' = v-\varepsilon\langle \varepsilon, v-w\rangle, \qquad w' = w+\varepsilon\langle \varepsilon, v-w\rangle \tag{1.3}$$

The function $n(t, x) = \int_{R^3} f(t, x, v)\, dv$ is the local density of the gas.

The way in which the geometric factor Y depends on x, $x \pm a\varepsilon$, and n gives rise to the different models of the Enskog equation found in the literature. In the original Enskog equation Y is given in terms of the equilibrium pair correlation function g_2, which depends on the local density only at the point of contact, i.e., $Y = g_2(n(t, x \pm \frac{1}{2}a\varepsilon))$. In the revised Enskog equation, Y arises as the pair correlation function g_2 for a system in which, at any time, the only correlations are due to the excluded volume of the spheres. In particular, there are no correlations between velocities in the system. In this case one can write $Y(x, x \pm a\varepsilon, n) = g_2(x, x \pm a\varepsilon \,|\, n(t))$, where the notation $h(x \,|\, n(t))$ means that the quantity $h(x)$ is a functional of the local density $n(x, t)$. The term "revised" points to the fact that in the revised Enskog equation g_2 corresponds to an inhomogeneous rather than a homogeneous equilibrium state. In terms of the formal Mayer cluster expansion, g_2 has the form[6]

$$g_2(x_1, x_2 \,|\, n(t)) = \theta_{12}\left\{1+\sum_{k=3}^{\infty}\frac{1}{(k-2)!}\int dx_3 \cdots \int dx_k\, n(3)\cdots n(k)\, V(12\,|\,3\cdots k)\right\} \tag{1.4}$$

where $n(k) = n(t, x_k)$, $x_k \in R^3$, $V(12 \mid 3 \cdots k)$ is the sum of all graphs of k labeled points which are biconnected when the Mayer factor $f_{12} = \theta_{12} - 1$ is added, $\theta_{12} \equiv \theta(|x_1 - x_2| - a)$, and θ is the Heaviside step function.

Particular choices of the geometric factor Y in the original and the revised Enskog equation give rise to important differences between these two equations. The revised equation has an analog of the Boltzmann H-theorem. Indeed, Résibois[4] showed that $H(t)$ given by

$$H(t) = \sum_{N=0}^{\infty} \int d\Gamma^N \rho_N(t) \log[N!\, \rho_N(t)] \tag{1.5}$$

is nonincreasing in $t \geqslant 0$, where $\rho_N(t)$ is the approximate N-particle distribution function, and that (at least formally) the revised Enskog equation drives the gas confined in a box with periodic boundary conditions toward the absolute Maxwellian.

The function $H(t)$ given in (1.5) can be rewritten in the form (ref. 4, p. 600)

$$H(t) = \iint f(t, x, v) \log f(t, x, v)\, dv\, dx + H^v(t) \tag{1.6}$$

where $f(t, x, v)$ is the solution to (1.1), and the potential part $H^v(t)$ is given in terms of Résibois' grand canonical formalism, but, unfortunately, not explicitly in terms of $f(t, x, v)$ and Y. This inability to express H^v explicitly in terms of f and Y has been one stumbling block in obtaining an existence theorem for the Enskog equation. I shall show in Section 2 that this difficulty in utilizing the H-function can be overcome.

Finally, the revised Enskog equation has a set of collision invariants that, as in the case of the Boltzmann equation, correspond to the conservation of mass, momentum, and energy on the macroscopic level.

The plan of the paper is as follows. In Section 2 I derive basic identities for the problem and state the main existence results. In Section 3 I state and prove a convergence theorem for approximate solutions. The convergence theorem, as stated in Section 3, equally applies to the revised Enskog equation, the Boltzmann equation, and the Boltzmann–Enskog equation ($Y \equiv 1$). This result unifies the process of solving Enskog-like (or Boltzmann) equations through the use of a *single* Liapunov functional and is a generalization of the DiPerna–Lions method developed for the Boltzmann equation. In Section 4, after deriving *a priori* estimations for the approximate solutions of the truncated Enskog equation, I show that the assumptions of the convergence theorem are satisfied in the several cases covered by Theorems 2.1–2.3, thus completing the proof of the existence theorems stated in Section 2.

I end this section with a brief review of known existence theorems for the original and revised Enskog equation (see ref. 7 for a more detailed review). The first local in time existence theorem was obtained by Lachowicz.[8] A global in time existence theorem was obtained by Toscani and Bellomo[9] in the case of a perturbation of the vacuum. I showed[10] that the solution obtained in ref. 9 is actually a classical solution to (1.1) if the initial datum is smooth. Furthermore, the asymptotic behavior of solutions was obtained in ref. 10. All of the above results deal with the original Enskog equation, but with easy modifications can be extended to the revised Enskog equation. Cercignani[20] obtained global in time solutions for small initial data in L^1 and $Y \equiv 1$.

The quoted results fall in either of two categories: small initial data or local in time existence results. For large initial data, Cercignani[11] obtained global in time L^1-solutions in the case of one space dimension and $Y \equiv 1$. Arkeryd[12] considered the two-dimensional spatial case using a weak compactness argument in L^1, however, with the range of integration with respect to ε extended to the whole sphere S^2, together with the assumption that $Y \equiv 1$. Observe that the alteration in the range of integration to the whole sphere S^2 has a significant effect on the dynamics of the Enskog equation. In fact, the original Enskog equation and the revised equation, with integration over S^2_+, distinguish between forward and backward (time-reversed) collisions, while the Boltzmann equation and the alteration above, with integration over S^2, are symmetric under forward and backward collisions. Recently, Arkeryd[13] has obtained a global existence for $Y \equiv 1$ under the assumptions that the initial value is differentiable in x in L^1 sense and has sufficiently high moments.

2. BASIC A PRIORI ESTIMATIONS AND FORMULATION OF THE EXISTENCE RESULTS

I indicated in the previous section of the form of the geometric factor for the revised Enskog equation. Due to the symmetry of $g_2(x_1, x_2 \mid n(t))$ in x_1 and x_2, given by formula (1.4), the revised Enskog collision operator has an analog of the H-function and a set of collision invariants corresponding to the conservation of mass, momentum, and energy. These two properties play a fundamental role in the existence theorems presented in this work. Below I state conditions for Y that will be used throught the paper. They not only imply the properties of the revised Enskog collision operator mentioned above, but also enlarge significantly the range of possible choices for Y.

I consider Y to be an arbitrary functional of the distribution function

f, as well possibly as an explicit function of the configuration space variables. Following conventional notation, write

$$Y \equiv Y(t, x_1, v_1, x_2, v_2 \,|\, Af(t)) \tag{2.1}$$

where, for each fixed $t \geqslant 0$, A indicates an operator, possibly nonlinear, acting on f, and $|\,Af(t)$ denotes the functional dependence of Y on $Af(t)$. I will assume throughout the paper that A and $|\,Af(t)$ act in such a way that Y in (2.1) is symmetric under the exchange of variables $x_1, v_1 \rightleftharpoons x_2, v_2$, and that Y is nonnegative for $f \geqslant 0$. From now on such Y will be called *symmetric*. It is clear that with A equal to the zeroth moment of the distribution function f, and functional dependence as in formula (1.4), we recover the revised Enskog equation. With the same A, I considered[14] Y that closely resembled the original Enskog equation. In that case functional dependence was such that Y reduced to a function of the local density at x_1 and x_2, i.e., $Y = Y(n(t, x_1), n(t, x_2))$. In both cases there is no dependence on velocities v_1 and v_2 in (2.1). Another model can be obtained by choosing A to be one of the higher moments of f, or more generally, to be of the following form: $Af = \int \phi(x, v)\, f(t, x, v)\, dv$. Here, for technical reasons, ϕ must be such that $|\phi(x, v)| \leqslant \text{const}(1 + |v|^k + |x|^k)$ for some $k < 2$, i.e., the second moment of f is not allowed as A. An important new generalization is obtained when one allows dependence on v_1 and v_2 in Y. In this case Y, as given by (2.1), resembles the exact two-particle correlation function for a hard-sphere gas. Equation (1.1) with Y given by (2.1) can be rewritten in the form

$$\frac{\partial f}{\partial f} + v\frac{\partial f}{\partial x} = E(f) \equiv E^+(f) - E^-(f) \tag{2.2}$$

with

$$E^+(f) = \iint_{R^3 \times S^2_+} Y(t, x, v', x - a\varepsilon, w' \,|\, Af(t)) \\ \times f(t, x, v')\, f(t, x - a\varepsilon, w') \langle \varepsilon, v - w \rangle \, d\varepsilon \, dw \tag{2.3a}$$

$$E^-(f) = \iint_{R^3 \times S^2_+} Y(t, x, v, x + a\varepsilon, w \,|\, Af(t)) \\ \times f(t, x, v)\, f(t, x + a\varepsilon, w) \langle \varepsilon, v - w \rangle \, d\varepsilon \, dw \tag{2.3b}$$

Equation (2.2) constitutes a basis for more general kinetic theory (see ref. 15; here, I consider only the hard-core potential). I will show that such a generalized Enskog equation (2.2) possesses a Liapunov functional. This new result makes (2.2) more attractive in spite of the fact that the explicit form of Y is unknown. Note that knowledge of the exact two-particle correlation function and the one-particle distribution function is equivalent to knowledge of the two-particle distribution function.

Finally, I give a last example of Y that is incorporated in the form (2.1). Let W be a function from $R^+ \times (R^3)^4 \times R \times R$ into R. For a fixed $t \geqslant 0$, define Y to be

$$Y \equiv W(t, x_1, v_1, x_2, v_2, Kf(t)(x_1, v_1), Kf(t)(x_2, v_2)) \tag{2.4}$$

where K is an operator, possible nonlinear, acting on f. In this case $\Lambda = K$ and the functional dependence is reduced to the action on the arguments of the function W. As in (2.1), W is such that Y is symmetric under the exchange of variables $x_1,\ v_1 \rightleftharpoons x_2,\ v_2$. The above examples indicate the large set of possible choices for Y which can be incorporated in formula (2.1).

Now I derive basic properties of (2.2). The first property of $E(f)$ is an analog of the corresponding identity for the Boltzmann collision operator. Identities similar to (35)–(37) of ref. 4 together with the fact that Y is *symmetric* imply that for ψ measurable on $R^3 \times R^3$ and $f \in C_0(R^3 \times R^3)$ we have

$$\begin{aligned}
&\iint_{R^3 \times R^3} \psi(x, v)\, E(f)\, dv\, dx \\
&\quad = \frac{1}{2} \iiiint_{R^3 \times R^3 \times R^3 \times S^2_+} [\psi(x, v') + \psi(x + a\varepsilon, w') - \psi(x,v) - \psi(x + a\varepsilon, w)] \\
&\qquad \times f(t, x, v)\, f(t, x + a\varepsilon, w)\, Y(t, x, v, x + a\varepsilon, w \mid \Lambda f(t)) \\
&\qquad \times \langle \varepsilon, v - w \rangle\, d\varepsilon\, dw\, dv\, dx
\end{aligned} \tag{2.5}$$

Observe that, except for the velocity dependence in Y, (2.5) with $\psi = \log f(t, x, v)$ is identity (37) in ref. 4.

For f a nonnegative solution of (2.2), and ignoring at this stage any integrability conditions, define

$$\Gamma(t) = \iint_{R^3 \times R^3} f(t, x, v) \log f(t, x, v)\, dv\, dx - \int_0^t I(s)\, ds \tag{2.6a}$$

where

$$\begin{aligned}
I(t) = \frac{1}{2} \iiiint_{R^3 \times R^3 \times R^3 \times S^2_+} &[f(t, x - a\varepsilon, w)\, Y(t, x, v', x - a\varepsilon, w' \mid \Lambda f(t)) \\
&- f(t, x + a\varepsilon, w)\, Y(t, x, v, x + a\varepsilon, w \mid \Lambda f(t))] \\
&\times f(t, x, v) \langle \varepsilon, v - w \rangle\, d\varepsilon\, dw\, dv\, dx
\end{aligned} \tag{2.6b}$$

Now, multiplying (2.2) by $1+\log f$ and integrating over $(x, v) \in R^3 \times R^3$ gives

$$\frac{d\Gamma}{dt} = \iint_{R^3 \times R^3} E(f) \log f \, dv \, dx - I(t) \tag{2.7}$$

Next, using (2.5) with $\psi = \log f$ together with the inequality $y(\log y - \log z) \geqslant y - z$ for $y, z > 0$ yields

$$\frac{d\Gamma}{dt} \leqslant 0 \tag{2.8}$$

The inequality (2.8) shows that $\Gamma(t)$ is a Liapunov functional for (2.2). $\Gamma(t)$ displays the dissipativity of the system governed by the generalized Enskog equation. It also can be considered as the analog of the H-function for (2.2). Note that in the dilute gas limit, when (2.2) becomes the Boltzmann equation, the function $\Gamma(t)$ reduces to the Boltzmann H-function. Furthermore, Γ is defined explicitly in terms of $f(t, x, v)$ and Y, in contrast to the H-function (1.5) [or (1.6)] obtained by Résibois.[4] Also, the fact that (2.8) is obtained without differentiation with respect to x or use of the continuity equation is crucial in the proof of the existence theorems. Indeed, the continuity equation used by Résibois[4] (see also refs. 5 and 15) in obtaining (2.8) for the original Liapunov functional cannot be applied to the various approximations of the Enskog collision operator that are needed in this work.

The last identity can be obtained by multiplying (2.2) by $(x - tv)^2$, integrating by parts over $x \in R^3$, and using (2.5) with $\psi = (x - tv)^2$ along with the equality

$$\begin{aligned}(x - tv')^2 &+ (x + a\varepsilon - tw')^2 \\ &= (x - tv)^2 + (x + a\varepsilon - tw)^2 - 2at\langle \varepsilon, v - w \rangle\end{aligned} \tag{2.9}$$

for $x, v, w \in R^3$, $t \in R$, $a > 0$, $\varepsilon \in S^2_+$, and v', w' given in (1.3). The result is

$$\begin{aligned}\frac{d}{dt} &\iint_{R^3 \times R^3} (x - tv)^2 f(t, x, v) \, dv \, dx \\ &= -at \iiiint_{R^3 \times R^3 \times R^3 \times S^2_+} \langle \varepsilon, v - w \rangle^2 \, Y(t, x, v, x + a\varepsilon, w \mid \Lambda f(t)) \, f(t, x, v) \\ &\quad \times f(t, x + a\varepsilon, w) \, d\varepsilon \, dw \, dv \, dx\end{aligned} \tag{2.10}$$

In view of (2.10), the functional defined by

$$\mathscr{E}(t) = \iint_{R^3 \times R^3} (x - tv)^2 f(t, x, v) \, dv \, dx \tag{2.11}$$

is another functional that indicates dispativity of the system. In the case of the Boltzmann equation, $a=0$ and $\mathscr{E}(t)=\mathscr{E}(0)$ for all $t\in R$. Note that $(d/dt)\,\mathscr{E}(t)<0$ only for positive t and nonnegative f. In addition, since identity (2.10) is true only for the whole space problem, $\mathscr{E}(t)$ may not be nonincreasing in the case of bounded spatial domain with appriopriate boundary conditions. Finally, since $\mathscr{E}(t)$ is decreasing for all times, solutions of (2.2), for the whole space problem, cannot approach an absolute Maxwellian.

Identity (2.5) and inequality (2.8) are crucial in the proof of the existence of global solutions of (2.2). One says that a nonnegative $f\in L^1_{\text{loc}}((0, T)\times R^3\times R^3)$ is a mild solution of (2.2) if, for each $0<T<\infty$, $E^{\pm}(f)(\cdot, x, v)^{\#}\in L^1(0, T)$ a.e. in $(x, v)\in R^3\times R^3$ and satisfies

$$f^{\#}(t, x, v)-f^{\#}(s, x, v)=\int_s^t E(f)^{\#}\,(\tau, x, v)\,d\tau, \qquad 0<s<t\leqslant T \tag{2.12}$$

Here $f^{\#}(t, x, v)=f(t, x+tv, v)$. Another definition of a solution, introduced by DiPerna and Lions[1] for the Boltzmann equation, deals with the notion of a so-called renormalized solution. One says that a nonnegative $f\in L^1_{\text{loc}}((0, T)\times R^3\times R^3)$ is a renormalized solution of (2.2) if

$$\frac{1}{1+f}\,E^{\pm}(f)\in L^1_{\text{loc}}((0, T)\times R^3\times R^3)$$

and

$$\frac{\partial}{\partial t}\log(1+f)+v\,\frac{\partial}{\partial x}\log(1+f)=\frac{1}{1+f}\,E(f) \tag{2.13}$$

in $\mathscr{D}'((0, \infty)\times R^3\times R^3)$. One can show, in the same way as in the case of the Boltzmann equation (see Lemma II.1 of ref. 1) that f is a renormalized solution of (2.2) if and only if it is a mild solution and

$$\frac{1}{1+f}\,E^{\pm}(f)\in L^1_{\text{loc}}((0, T)\times R^3\times R^3)$$

Finally, let $F^{\#}(t, x, v)=\int_0^t L^{+}(f)^{\#}\,(\tau, x, v)\,d\tau$, where $L^{+}(f)$ is defined by $E^{-}(f)=fL^{+}(f)$. If $L^{+}(f)\in L^1_{\text{loc}}((0, T)\times R^3\times R^3)$ for any $T>0$, then f is a mild solution of (2.2) if and only if f satisfies

$$\begin{aligned} &f^{\#}(t, x, v)-f^{\#}(s, x, v)\exp\{-[F^{\#}(t)-F^{\#}(s)]\} \\ &\qquad=\int_s^t E^{+}(f)^{\#}\,(\tau, x, v)\exp\{-[F^{\#}(t)-F^{\#}(\tau)]\}\,d\tau \end{aligned} \tag{2.14}$$

for any $0<s<t\leqslant T$ and a.e. in $(x, v)\in R^3\times R^3$.

Before stating the existence results for (2.2), I impose additional regularity conditions on Y. I start with several definitions. For $M>0$ define the set $D_M=\{f\in L^1(R^3\times R^3)\colon f\geqslant 0,\ \iint_{R^3\times R^3}(1+v^2)f\,dv\,dx\leqslant M\}$. One says that Y, given by (2.1), is *bounded* if, for each $T>0$, $M>0$, and $f\in D_M$, $Y(t,x,v,x+a\varepsilon,w\,|\,Af)$ is measurable in $(t,x,v,w,\varepsilon)\in[0,T]\times R^3\times R^3\times R^3\times S^2_+\equiv W$, and $\sup_{f\in D_M}\{\|Y\|_{L^\infty(W)}\}\leqslant C(M,T)<\infty$. Finally, a *symmetric* and *bounded* Y is *regular* if there exists a sequence $\{Y_k\}$ of approximations of Y such that the following conditions are satisfied:

(i) For each $k\geqslant 1$, Y_k is *symmetric* and *bounded*, and for each f, $g\in D_M$,

$$\begin{aligned}\iint_{\Omega_k\times S^2_+}\sup_{\substack{(v,w)\in B_k\\ t\in[0,T]}}&|Y_k(t,x,v,x+a\varepsilon,w\,|\,Af)\\&-Y_k(t,x,v,x+a\varepsilon,w\,|\,Ag)|\,dx\,d\varepsilon\\&\leqslant C(k,M,T)\,\|f-g\|_{L^1}\end{aligned}\tag{2.15}$$

where $B_k=\{(v,w)\in R^3\times R^3\colon v^2+w^2\leqslant k\}$ and $\Omega_k=\{x\in R^3\colon |x|\leqslant k\}$.

(ii) For any L^1-weakly compact sequence $\{f_k\}\subset L^1((0,T),D_M)$ with the property that the set $\{\int_{R^3}\varphi f_k\,dv\}_{k=1}^\infty$ is compact in $L^1((0,T)\times R^3)$ for each $\varphi\in L^\infty((0,T)\times R^3\times R^3)$, there is a subsequence $\{f_{k_i}\}$ such that

$$Y_{k_i}(t,x,v,x\pm a\varepsilon,w\,|\,Af_{k_i}(t))\xrightarrow[i\to\infty]{}Y(t,x,v,x\pm a\varepsilon,w\,|\,Af(t))\quad\text{a.e. in}\quad t,x,v,w,\varepsilon\tag{2.16}$$

Assumption (2.15) guarantees the existence of solutions for a suitable truncated problem that will be considered in Section 4. On the other hand, pointwise convergence in (2.16) is needed in the proof the convergence theorem given in Section 3.

Note that there exists a large class of Y that are *regular*. Indeed, Y defined by (2.4) with $K=\int\phi(x,v)\,f(t,x,v)\,dv$ and $|\phi(x,v)|\leqslant\text{const}(1+|v|^\lambda+|x|^\lambda)$ for some $\lambda<2$ is *regular* if W in (2.4) is any L^∞-function that is *symmetric*. To see this, it is enough to notice that by Lusin's theorem, $W\circ R_k$ can be pointwise approximated by C_c^∞ functions W_{i_k}. Here, R_k, applied to each argument of W, is the radial retraction, i.e., $R_k(r)=r$ for $r\leqslant k$ and $R_k(r)=k(r/|r|)$ for $r>k$. Since $|R_k(r_1)-R_k(r_2)|\leqslant 2\,|r_1-r_2|$ for any $k\geqslant 1$, one easily obtains condition (2.15) for W_{i_k} with the operator K replaced by $K_k=\int_{|v|\leqslant k}\phi(x,v)\,f(t,x,v)\,dv$. Finally, since $R_k(r)\xrightarrow[k\to\infty]{}r$ and $\lambda<2$, we obtain convergence in (2.16). In particular, the above arguments also imply that Y considered in ref. 14, i.e., $Y=Y(n(t,x_1),n(t,x_2))$ with Y measurable, bounded, and symmetric as a function of two variables, is *regular*. Furthermore, the operator K considered above can be replaced by any weakly compact linear operator in $L^1(R^3 \times R^3)$ (see Chapter 9 of ref. 16 for more details on weakly compact operators).

Also note that with the notation as in (1.4) and for any $i \geqslant 3$

$$Y = \left| \theta_{12} \left\{ 1 + \sum_{k=3}^{i} \frac{1}{(k-2)!} \int dx_3 \cdots \int dx_k \, n(3) \cdots n(k) \, V(12 \mid 3 \cdots k) \right\} \right| \tag{2.17}$$

is *regular*. At this time I do not know if the same is true for $i = \infty$.

Theorem 2.1. Suppose that Y is regular and $f_0 \geqslant 0$ satisfies

$$\iint_{R^3 \times R^3} (1 + v^2 + x^2 + |\log f_0|) f_0 \, dv \, dx = C_0 < \infty \tag{2.18}$$

If either (1) $T > 0$ is arbitrary and $\|f_0\|_{L^1(R^3 \times R^3)}$ is small enough or (2) $T > 0$ is small enough, then there exists a mild solution $f(t, x, v)$ of (2.2) such that $\lim_{t \to 0^+} f(t, x, v) = f_0(x, v)$ a.e. in $(x, v) \in R^3 \times R^3$.

Another important existence result can be obtained if the scattering kernel $\langle \varepsilon, v - w \rangle$ in $E(f)$ is replaced by

$$\chi_\gamma \times \langle \varepsilon, v - w \rangle \tag{2.19}$$

where χ_γ is the characteristic function of the set $\{(\varepsilon, v, w) \in S_+^2 \times R^3 \times R^3\colon \langle \varepsilon, v - w \rangle \geqslant \gamma\}$ and $\gamma > 0$ is arbitrarily small. From a physical point of view it means that we eliminate collisions (called the grazing collisions) that result in small changes of v' and w' as compared with their precollisional values v and w, respectively. Note that a similar cutoff has been common in the case of the Boltzmann collision operator. Indeed, the restriction of the deflection angle θ to $0 \leqslant \theta \leqslant \pi/2 - \gamma$ for some small $\gamma > 0$ results in elimination of the grazing collisions. Note, however, that the angular cutoff in the case of the Boltzmann equation was needed to handle a singularity resulting from an infinite range of interactions of the inverse power potentials. Here, since we consider only hard spheres, such a singularity does not appear. For technical reasons, however, we still need the truncation as in (2.19).

We have the following result.

Theorem 2.2. Suppose that Y is regular and $f_0 \geqslant 0$ satisfies (2.18). Then for any $\gamma > 0$ there exists a global in time mild solution of the generalized Enskog equation with the scattering kernel given in (2.19).

The final existence theorem deals with the case of Y considered in ref. 14:

Theorem 2.3. Suppose that Y is regular and $f_0 \geqslant 0$ satisfies (2.18). In addition, assume that $Y = Y(n(t, x_1), n(t, x_2))$ with Y measurable, bounded, and symmetric as a function of two variables has a compact support as a function of two variables. Then there exists a global in time mild solution $f(t, x, v)$ of (2.2) such that $\lim_{t \to 0^+} f(t, x, v) = f_0(x, v)$ a.e. in $(x, v) \in R^3 \times R^3$.

Theorem 2.3 also holds for bounded spatial domains with periodic boundary conditions. In this case it is convenient to include boundary conditions in the definition of the spatial domain by considering Ω a three-dimensional torus, i.e., $\Omega = R^3/Z^3$. Mild and renormalized solutions of (2.2) in this case are defined in an analogous way as in the whole-space problem.

The proofs of the above theorems, provided at the end of Section 4, are based on the convergence result given in Section 3 (Theorem 3.2) as well as on the *a priori* estimations for the approximate solutions derived in Section 4.

3. CONVERGENCE OF APPROXIMATE SOLUTIONS

The process of solving (2.2) is divided into several steps. The idea is to find a suitable truncated version E_n of E for which (2.2) can be solved for each $n \geqslant 1$. The E_n should retain all basic properties of E, in particular, identities (2.5)–(2.7) together with inequality (2.8). In this section I show that under a certain condition [see (3.9)] on the Liapunov function (2.6), a sequence of solutions to the truncated Enskog equation converges to a mild solution of (2.2). Later, in Section 4, I actually prove that (3.9) is satisfied in the several cases covered by Theorems 2.1–2.3.

Consider a nonnegative solution f_n to the initial value problem

$$\frac{\partial f_n}{\partial t} + v \frac{\partial f_n}{\partial x} = E_n(f_n) \equiv E_n^+(f_n) - E_n^-(f_n), \quad f_n(0, x, v) = f_0(x, v), \quad 0 < t \leqslant T \tag{3.1}$$

E_n is defined to be the Enskog operator on the right-hand side of (2.2) with $\langle \varepsilon, v - w \rangle$ replaced by $\langle \varepsilon, v - w \rangle \times W_n$. The Y appearing in E^+ and E^- of (2.3a) and (2.3b) are replaced by $Y_n^- \times X_n^- [x^2 + (x - a\varepsilon)^2]$ and $Y_n^+ \times X_n^+ [x^2 + (x + a\varepsilon)^2]$, respectively, where $X_n^{\pm}(z) = 1$ for $|z| \leqslant n^2$ and $X_n^{\pm}(z) = 0$ otherwise. Here $Y_n^{\pm} = Y_n(t, x, v, x \pm a\varepsilon, w \mid \Lambda f_n(t))$, where Y_n is symmetric, bounded, and satisfies (2.16). Finally, $W_n =$

$[\cos\theta(1/n+\cos\theta)^{-1}]\,\chi_n$, where $\chi_n = 1$ if $v^2+w^2 \leqslant n$ and $\chi_n = 0$ otherwise, and $\cos\theta = \langle v-w, \varepsilon\rangle/|v-w|$.

In order to obtain *a priori* estimations for the solution $f_n(t, x, v)$ of (3.1), I assume throughout this section that f_n is a smooth and nonnegative solution with the initial value $f_0 \geqslant 0$ satisfying

$$\iint_{R^3\times R^3} (1+v^2+x^2+|\log f_0(x, v)|)\, f_0(x, v)\, dv\, dx \leqslant C_0 < \infty \tag{3.2}$$

The first *a priori* estimations are the following conservation laws, which follow from (2.5) with $\psi = 1, v, v^2$:

$$\iint_{R^3\times R^3} f_n(t, x, v)\, dv\, dx = \iint_{R^3\times R^3} f_0(x, v)\, dv\, dx \tag{3.3}$$

$$\iint_{R^3\times R^3} v f_n(t, x, v)\, dv\, dx = \iint_{R^3\times R^3} v f_0(x, v)\, dv\, dx \tag{3.4}$$

$$\iint_{R^3\times R^3} v^2 f_n(t, x, v)\, dv\, dx = \iint_{R^3\times R^3} v^2 f_0(x, v)\, dv\, dx \tag{3.5}$$

Next, using (2.10) together with the Cauchy–Schwarz inequality applied to

$$\iint_{R^3\times R^3} \langle x\sqrt{f_n}, v\sqrt{f_n}\rangle\, dv\, dx$$

gives, uniformly in $n \geqslant 1$,

$$\sup_{t\in[0,T]} \iint_{R^3\times R^3} x^2 f_n(t, x, v)\, dv\, dx \leqslant C_1$$

where C_1 depends on T, on $\iint_{R^3\times R^3} x^2 f_0\, dv\, dx$, and on $\iint_{R^3\times R^3} (1+v^2)\, f_0\, dv\, dx$.

Combining all the above, one has for initial data satisfying (3.2)

$$\sup_{\substack{t\in[0,T]\\ n\geqslant 1}} \iint_{R^3\times R^3} (1+v^2+x^2)\, f_n(t, x, v)\, dv\, dx \leqslant C_T \tag{3.6}$$

where C_T depends on T and f_0.

Let $I_n^{\pm}(t)$ be defined by

$$\begin{aligned} I_n^{\pm}(t) \equiv \frac{1}{2} \iiiint_{R^3\times R^3\times R^3\times S_+^2} & f_n(t, x\mp a\varepsilon, w) \\ & \times Y_n^{\mp} X_n^{\mp} f_n(t, x, v) \langle\varepsilon, v-w\rangle\, W_n\, d\varepsilon\, dw\, dv\, dx \end{aligned} \tag{3.7}$$

Then, integration of (2.8) with respect to t yields

$$\iint_{R^3\times R^3} f_n(t,x,v)\log^+ f_n(t,x,v)\,dv\,dx+\int_0^t I_n^-(s)\,ds$$
$$\leqslant \iint_{R^3\times R^3} f_n(t,x,v)\log^- f_n(t,x,v)\,dv\,dx+\int_0^t I_n^+(s)\,ds+E_0 \tag{3.8}$$

for all $t\in[0,T]$, where

$$E_0=\iint_{R^3\times R^3} f_0\,|\log f_0|\,dv\,dx$$
$$\log^{\pm}(z)\equiv\max\{\pm\log(z),0\}$$

Next, use of the inequality $z\log(z/y)\geqslant -y$ with $y=\exp(-x^2-v^2)$ and $z=f_n$ together with estimation (3.6) shows that the first term on the right-hand side of (3.8) is bounded uniformly in $n\geqslant 1$ and $t\in[0,T]$. Since, by assumption (3.2), $E_0<\infty$, one obtains that

$$\sup_{0\leqslant t\leqslant T}\iint_{R^3\times R^3} f_n(t,x,v)\log^+ f_n(t,x,v)\,dv\,dx$$

is bounded uniformly in $n\geqslant 1$ as long as

$$\sup_{n\geqslant 1}\int_0^T I_n^+(s)\,ds\leqslant C_2(C_0,T)<\infty \tag{3.9}$$

Summarizing all the above, it can be concluded that, for initial data satisfying (3.2), bound (3.9) implies that

$$\sup_{\substack{t\in[0,T]\\ n\geqslant 1}}\iint_{R^3\times R^3}(1+v^2+x^2+|\log f_n(t,x,v)|)\,f_n(t,x,v)\,dv\,dx\leqslant C_T \tag{3.10}$$

where C_T depends only on T and f_0.

Estimation (3.10) places the Enskog equation (2.2) in the framework of the DiPerna–Lions method developed for the Boltzmann equation. Observe that condition (3.9) is superfluous in the case of the Boltzmann equation. Indeed, $a=0$ and $Y=1$ imply that $I_n^+(t)\equiv I_n^-(t)$ [i.e., $I_n(t)\equiv 0$ in (2.6b)] for $t\in[0,T]$ and $n\geqslant 1$. Hence the bound (3.10) can be obtained directly from inequality (3.8). In fact, in the case of the Boltzmann equation we also have $Y=Y_n=Y_n^-=Y_n^+\equiv 1$.

Note that for Y independent of velocities, and for the Enskog equation with integration with respect to ε extended to the whole sphere S^2, the

Liapunov functional $\Gamma(t)$, given in (2.6a), reduces to the Boltzmann H-function. Indeed, integration over the whole sphere S^2 implies that $I(t)$ defined in (2.6b) vanishes, i.e., $I(t) \equiv 0$. Thus, in this case, too, condition (3.9) does not play any role and this equation, except for the multiplicative factor Y, is equivalent to the Boltzmann equation.

Also remark that in the case of one space dimension condition (3.9) is satisfied (see ref. 14, p. 169) for any Y that is *bounded.*

Finally, observe that if one knows that the local density $n(t, x)$ is bounded, then condition (3.9) is always satisfied. In spite of the fact that finiteness of $n(t, x)$ is expected for the system of hard spheres, it is not clear that this indeed is a property of the generalized or the revised Enskog equations.

Next, I show that condition (3.9) implies also an important gain–loss estimation. This estimation, together with (3.10), is fundamental in applying the weak compactness argument to the Enskog equation. First, one has, for each $M > 1$,

$$E_n^+(f_n) \leqslant M \iint_{R^3 \times S_+^2} Y_n^- X_n^- f_n(t, x, v) f_n(t, x - a\varepsilon, w) \langle \varepsilon, v - w \rangle W_n \, d\varepsilon \, dw + \frac{1}{\log M} \alpha(f_n) \tag{3.11}$$

where

$$\alpha(f_n) = \iint_{R^3 \times S_+^2} Y_n^- X_n^- f_n(t, x, v') f_n(t, x - a\varepsilon, w') \times \left| \log \frac{f_n(t, x, v') f_n(t, x - a\varepsilon, w')}{f_n(t, x, v) f_n(t, x - a\varepsilon, w)} \right| \langle \varepsilon, v - w \rangle W_n \, d\varepsilon \, dw$$

Multiplying (3.1) by $1 + \log f_n$, integrating over $(x, v) \in R^3 \times R^3$, and using (2.5) with $\psi = \log f_n$ gives

$$\iint_{R^3 \times R^3} f_n(t, x, v) \log f_n(t, x, v) \, dv \, dx - \iint_{R^3 \times R^3} f_0 \log f_0 \, dv \, dx = \int_0^t \iiiint_{R^3 \times R^3 \times R^3 \times S_+^2} h(f_n) \, d\varepsilon \, dw \, dv \, dx \, ds \tag{3.12}$$

for

$$h(f_n) = -\frac{1}{2} Y_n^- X_n^- f_n(s, x, v') f_n(s, x - a\varepsilon, w') \times \log \frac{f_n(s, x, v') f_n(s, x - a\varepsilon, w')}{f_n(s, x, v) f_n(s, x - a\varepsilon, w)} \langle \varepsilon, v - w \rangle W_n$$

The inequality $z(\log z - \log y) \geqslant z - y$ implies that for $h^+(f_n) = \max\{h(f_n), 0\}$ one has

$$\iiiint\!\!\int_{[0,T]\times R^3\times R^3\times R^3\times S^2_+} h^+(f_n)\, d\varepsilon\, dw\, dv\, dx\, ds \leqslant \sup_{n\geqslant 1} \int_0^T I_n^+(s)\, ds \tag{3.13}$$

Finally, (3.2) and (3.10) imply that the left-hand side of (3.12) is bounded. Hence, for $h^-(f_n) = \max\{-h(f_n), 0\}$, one has

$$\iiiint\!\!\int_{[0,T]\times R^3\times R^3\times R^3\times S^2_+} h^-(f_n)\, d\varepsilon\, dw\, dv\, dx\, ds \leqslant \text{const}(C_0, C_T)$$

Since

$$\begin{aligned}&\iiint_{[0,T]\times R^3\times R^3} \alpha(f_n)\, dv\, dx\, ds\\ &\quad = \iiiint\!\!\int_{[0,T]\times R^3\times R^3\times R^3\times S^2_+} 2[h^+(f_n) + h^-(f_n)]\, d\varepsilon\, dw\, dv\, dx\, ds\end{aligned}$$

it can be concluded that under condition (3.9)

$$\sup_{n\geqslant 1} \|\alpha(f_n)\|_{L^1((0,T)\times R^3\times R^3)} \leqslant C(C_0, C_T) < \infty \tag{3.14}$$

Inequality (3.11) is an analog of the gain–loss estimation for the Boltzmann collision operator.[1] Note that the first term on the right-hand side of (3.11) is not exactly the loss term of $E_n(f_n)$. Furthermore, in the case of the Boltzmann equation, $\alpha(f_n)$ in (3.11) is replaced by

$$\begin{aligned}\alpha_B(f_n) = \iint_{R^3\times S^2_+} &[f_n(t, x, v')\, f_n(t, x, w') - f_n(t, x, v)\, f_n(t, x, w)]\\ &\times \log \frac{f_n(t, x, v')\, f_n(t, x, w')}{f_n(t, x, v)\, f_n(t, x, w)}\, B(\theta, v - w)\, d\varepsilon\, dw\end{aligned}$$

Since $\alpha_B(f_n) \geqslant 0$ and identity (3.12) holds with $h(f_n)$ replaced by $-\alpha_B(f_n)$, one obtains (3.14) for $\alpha_B(f_n)$ directly from (3.12). Similar simplification of the gain–loss estimation is achieved for the Enskog equation with integration in ε performed over the whole S^2. However, for the Enskog equation with its full dynamics, (3.9) was needed in order to obtain a uniform L^1-bound on $\alpha(f_n)$.

The last ingredient needed for convergence of f_n to a mild solution of (2.2) is a compactness result due to Golse *et al.*,[17] which applies to general transport equations. Below I state a version directly applicable in the present setting (see Corollary IV.1 of ref. 1).

Lemma 3.0. Suppose that f_n, $g_n \in L^1_{\text{loc}}((0, T) \times R^3 \times R^3)$ satisfy

$$T_v f_n \stackrel{\text{def}}{=} \frac{\partial f_n}{\partial t} + v \frac{\partial f_n}{\partial x} = g_n \tag{3.15}$$

in $\mathscr{D}'((0, T) \times R^3 \times R^3)$, and for each compact set K of $(0, T) \times R^3 \times R^3$, the sequences $\{f_n\}$ and $\{g_n\}$ are weakly compact in $L^1((0, T) \times R^3 \times R^3)$ and $L^1(K)$, respectively. Then for all $\varphi \in L^1((0, T) \times R^3 \times R^3)$ the set $\{\int_{R^3} \varphi f_n \, dv\} = \{\int_{R^3} \varphi T_v^{-1} g_n \, dv\}$ is compact in $L^1((0, T) \times R^3)$.

In other words, the velocity averaged operator T_v^{-1} behaves in a similar way to the inverse of an elliptic operator. Recall that T_v^{-1} may be singular only on the set of the characteristic direction. Velocity averaging compensates for the lack of regularity in the characteristic direction of the hyperbolic operator T_v.

Finally, the following simple lemma will be used in the proof of Theorem 3.2 (see Theorem 4.21.10, p. 279 of ref. 16).

Lemma 3.1. Assume that g_n and h_n are measurable on $\Sigma = (0, T) \times R^3 \times R^3$. We have:

(i) $\int_\Sigma h_n g_n \to \int_\Sigma hg$ if $g_n \to g$ weakly in L^1, $\sup_{n \geqslant 1} \|h_n\|_{L^\infty} < \infty$, and $h_n \to h$ a.e.

(ii) $\int_\Sigma h_n g_n \to \int_\Sigma hg$ if $h_n \to h$ weakly in L^1, $g_n \to g$ strongly in L^1, and $\sup_{n \geqslant 1} \|h_n\|_{L^\infty} < \infty$.

Theorem 3.2. Let $\{f_n\}$ be a sequence of nonnegative and mild solutions of (3.1) with the initial value f_0 satisfying (3.2). In addition, assume that the following are satisfied:

(i) $\sup_{n \geqslant 1} \int_0^T I_n^+(s) \, ds \leqslant C_T < \infty$, where $I_n^+(s)$ is defined in (3.7)

(ii) $\sup_{t \in [0, T], n \geqslant 1} \iint_{R^3 \times R^3} [1 + v^2 + x^2 + |\log f_n(t, x, v)|] f_n(t, x, v) \, dv \, dx \leqslant C_T < \infty$

(iii) For each $n \geqslant 1$, $f_n \in L^\infty((0, T) \times R^3 \times R^3)$

Then there exists a subsequence $\{f_{n_i}\}$ converging weakly in $L^1((0, T) \times R^3 \times R^3)$ to a mild solution f of (2.2).

Assumption (i) of Theorem 3.2, i.e., condition (3.9) imposed on the Liapunov functional $\Gamma(t)$, was essential in deriving the *a priori* estimation (3.10) for *smooth* solutions of (3.1). Due to the form of E_n in Eq. (3.1), a solution f_n of the truncated problem may not be necessary smooth. In Theorem 3.2 we start with a mild solution of (3.1) and assume (ii). In Section 4 I will show how to derive (ii) from (i) for f_n not necessary smooth, and will prove (3.9) in the cases considered by Theorems 2.1–2.3.

Conditions (ii) and (iii) also imply that for each $n \geqslant 1$, $E_n^{\pm}(f_n) \in L^1 \cap L^\infty((0, T) \times R^3 \times R^3)$, thus making each f_n a renormalized solution of (3.1), as well as a solution of the exponential multiplier form (2.14).

Theorem 3.2 can be viewed as a stability result for the Enskog equation, as well as a convergence theorem for approximate solutions. As opposed to the DiPerna–Lions work[1] for the Boltzmann equation, I do not treat separately the cases for which the scattering kernel $B(\theta, |v-w|)$ is bounded integrable and unbounded nonintegrable, as a function of $v - w$ [$B(\theta, |v-w|) = \langle \varepsilon, v-w \rangle$ for the hard-sphere model].

The proof is preceded by several preliminary results. In the rest of this section $\{f_n\}$ denotes the sequence (or subsequence) of nonnegative and mild solutions of (3.1) given in Theorem 3.2 with properties (i)–(iii), $\Sigma \equiv (0, T) \times R^3 \times R^3$, and $\Sigma_R \equiv (0, T) \times R^3 \times B_R$, where $B_R = \{z \in R^3 \colon |z| \leqslant R\}$ for any $R > 0$. The proofs of convergence in (3.18) of the first proposition, as well as convergence in (3.32) of the following proposition, utilize a delicate argument based on the nonnegativity of f_n and φ. This is due to the fact that in the Enskog collision operator one of the spatial arguments is shifted to $x \pm a\varepsilon$.

Proposition 3.3. The sequence $\{f_n\}$ is weakly compact in $L^1(\Sigma)$ and there exists a nonnegative f such that

$$\sup_{t \in [0, T]} \iint_{R^3 \times R^3} [1 + v^2 + x^2 + |\log f(t, x, v)|]\, f(t, x, v)\, dv\, dx \leqslant C'_T < \infty \tag{3.16}$$

In addition, for each φ with $(1 + |x|^k + |v|^k)^{-1} \varphi \in L^\infty(\Sigma)$ and $0 \leqslant k < 2$, one has

$$\int_{R^3} f_n \varphi\, dv \xrightarrow[n \to \infty]{} \int_{R^3} f\varphi\, dv \qquad \text{in} \quad L^1((0, T) \times R^3) \tag{3.17}$$

and

$$L_n^{\pm}(f_n) \xrightarrow[n \to \infty]{} L^{\pm}(f) \qquad \text{in} \quad L^1(\Sigma_R) \tag{3.18}$$

for any $R > 0$, where

$$L_n^{\pm}(f_n) = \iint_{R^3 \times S_+^2} Y_n^{\pm} X_n^{\pm} f_n(t, x \pm \varepsilon, w) \langle \varepsilon, v - w \rangle\, W_n\, d\varepsilon\, dw$$

and

$$L^{\pm}(f) = \iint_{R^3 \times S_+^2} Y(t, x, v, x \pm \varepsilon, w \mid \Lambda f(t))\, f(t, x \pm \varepsilon, w) \langle \varepsilon, v - w \rangle\, d\varepsilon\, dw$$

Proof. Assumption (ii) of Theorem 3.2 and the Dunford–Pettis theorem (Theorem 4.21.2, p. 274 of ref. 16) imply that $\{f_n\}$ is weakly compact in $L^1(\Sigma)$. Without loss of generality, one may assume that $f_n \to f$ weakly in $L^1(\Sigma)$, where $0 \leqslant f \in L^1(\Sigma)$. By Fatou's lemma

$$\iint_{R^3 \times R^3} (1 + v^2 + x^2) f(t, x, v)\, dv\, dx \leqslant C_T^0 \qquad \text{a.e. in } t \in [0, T] \tag{3.19}$$

Now (3.16) will follow if it is shown that

$$\iint_{R^3 \times R^3} f \log f\, dv\, dx \leqslant \liminf_{n \to \infty} \iint_{R^3 \times R^3} f_n \log f_n\, dv\, dx \qquad \text{a.e. in } t \in [0, T] \tag{3.20}$$

However, since $z \log z$ is convex, lower weak semicontinuity of $\iint_{R^3 \times R^3} f \log f\, dv\, dx$ is equivalent to lower semicontinuity. Therefore, it is enough to show (3.20) for f_n converging strongly to f in $L^1(R^3 \times R^3)$. In addition, because of (ii) of Theorem 3.2, one can further assume that $\beta f_n \to \beta f$ in $L^1(R^3 \times R^3)$, where $\beta = 1 + |x| + |v|$. Since $f_n \log^- f_n \leqslant \beta f_n + \exp(-\beta)$, one sees that, by passing to a subsequence if necessary, $f_n \log^- f_n \to f \log^- f$ in $L^1(R^3 \times R^3)$. Hence,

$$\liminf_{n \to \infty} \iint_{R^3 \times R^3} f_n \log^+ f_n\, dv\, dx \leqslant c_1 + \iint_{R^3 \times R^3} f \log^- f\, dv\, dx$$

where $c_1 = \sup_{n \geqslant 1} \iint_{R^3 \times R^3} f_n \log f_n\, dv\, dx$. Fatou's lemma applied to the last inequality implies (3.20).

Now I prove (3.17). First, for each $\delta > 0$ and $R > 0$, assumption (ii) of Theorem 3.2 and the fact that the range of ε is a bounded set imply that the two sequences (corresponding to "$\pm$")

$$\left\{ \frac{1}{1 + \delta f_n} (P_\varepsilon^\pm f_n)\, Y_n^\pm X_n^\pm f_n \langle \varepsilon, v - w \rangle\, W_n \right\} \tag{3.21}$$

as weakly compact in $L^1(N)$ with $N \equiv (0, T) \times R^3 \times B_R \times R^3 \times S_+^2$. Here $(P_\varepsilon^\pm f_n)(t, x, w, \varepsilon) = f_n(t, x \pm a\varepsilon, w)$. By choosing the sequence in (3.21) corresponding to "+" together with the fact that the integral $\iint_{R^3 \times S_+^2} d\varepsilon\, dw$ is a bounded operator from $L^1(N)$ to $L^1(\Sigma_R)$, one obtains that for each $\delta > 0$ and $R > 0$ the sequence $\{E_n^-(f_n)/(1 + \delta f_n)\}$ is weakly compact in $L^1(\Sigma_R)$. By taking the sequence in (3.21) corresponding to "$-$" one obtains that the sequence $\{G_n(f_n)/(1 + \delta f_n)\}$ is weakly compact in $L^1(\Sigma_R)$, where

$$G_n(f_n) = \iint_{R^3 \times S_+^2} Y_n^- X_n^- f_n(t, x, v)\, f_n(t, x - a\varepsilon, w) \langle \varepsilon, v - w \rangle\, W_n\, d\varepsilon\, dw$$

Now the gain–loss estimation (3.11), together with bound (3.14), implies that for each $\delta > 0$ and $R > 0$ the sequence $\{E_n^+(f_n)/(1+\delta f_n)\}$ is weakly compact in $L^1(\Sigma_R)$.

Consider $f_n^\delta = (1/\delta)\log(1+\delta f_n)$ for $\delta > 0$ and observe that $0 \leqslant f_n^\delta \leqslant f_n$. Thus, the sequence $\{f_n^\delta\}_{n=1}^\infty$ is weakly compact in $L^1(\Sigma)$. Furthermore, f_n^δ satisfies

$$\frac{\partial}{\partial t} f_n^\delta + v \frac{\partial}{\partial x} f_n^\delta = \frac{1}{1+\delta f_n} E_n(f_n) \tag{3.22}$$

in $\mathscr{D}'((0, \infty) \times R^3 \times R^3)$. Therefore, Lemma 3.0 implies that for all $\varphi \in L^\infty(\Sigma)$ one has

$$\int_{R^3} f_n^\delta \varphi \, dv \xrightarrow[n \to \infty]{} \int_{R^3} f^\delta \varphi \, dv \qquad \text{in} \quad L^1((0, T) \times R^3) \tag{3.23}$$

where f^δ, after passing to subsequence if necessary, is the weak limit of $\{f_n^\delta\}$. I claim that (3.17) follows for $\varphi \in L^\infty(\Sigma)$ if it can be shown that

$$\sup_{n \geqslant 1} \sup_{t \in [0, T]} \|f_n - f_n^\delta\|_{L^1(R^3 \times R^3)} \xrightarrow[\delta \to 0^+]{} 0 \tag{3.24}$$

Indeed, since the norm is lower weakly semicontinuous, one obtains from (3.24)

$$\|f - f^\delta\|_{L^1(\Sigma)} \leqslant T \sup_{t \in [0, T]} \liminf_{n \to \infty} \|f_n - f_n^\delta\|_{L^1(R^3 \times R^3)} \xrightarrow[\delta \to 0^+]{} 0 \tag{3.25}$$

and

$$\int_{R^3} f_n \varphi \, dv = \int_{R^3} (f_n - f_n^\delta) \varphi \, dv + \int_{R^3} (f_n^\delta - f^\delta) \varphi \, dv + \int_{R^3} f^\delta \varphi \, dv \tag{3.26}$$

Now the application of (3.23)–(3.25) gives (3.17) for $\varphi \in L^\infty(\Sigma)$. In order to show (3.24), notice that

$$\begin{aligned} 0 &\leqslant s - \frac{1}{\delta} \log(1 + \delta s) \\ &\leqslant s \left[\left(1 - \frac{\log(1+\delta s)}{\delta s} \right) \chi_{\{s \leqslant R\}} \right] + s \chi_{\{s \geqslant R\}} \\ &= s \Theta_R(\delta) + s \chi_{\{s \geqslant R\}} \end{aligned} \tag{3.27}$$

where $\Theta_R(\delta) \xrightarrow[\delta \to 0^+]{} 0$ locally uniformly in R and χ_A denotes the characteristic function of A. Next, assumption (ii) of Theorem 3.2 implies that

$$\sup_{n \geqslant 1} \sup_{t \in [0, T]} \iint_{R^3 \times R^3} f_n \chi_{\{f_n \geqslant R\}} \, dv \, dx \xrightarrow[R \to \infty]{} 0$$

thus completing the proof of (3.17) for $\varphi \in L^\infty(\Sigma)$. Finally, combination of (3.16) together with the fact that $0 \leqslant k < 2$ gives (3.17) for φ with $(1+|x|^k+|v|^k)^{-1}\varphi \in L^\infty(\Sigma)$.

Now I proceed to the proof of (3.18). First, observe that

$$\iint_{R^3\times S^2_+} Y_f^\pm (P_\varepsilon^\pm f_n)\langle \varepsilon, v-w\rangle \, d\varepsilon \, dw \xrightarrow[n\to\infty]{} L^\pm(f) \qquad \text{in } L^1(\Sigma_R) \tag{3.28}$$

where $Y_f^\pm = Y(t, x, v, x\pm\varepsilon, w \mid \Lambda f(t))$. Indeed, since by (3.17) the sequence

$$\left\{ \iint_{S^3\times R^3} Y_f^\pm (P_\varepsilon^\pm f_n)\, |\langle \varepsilon, v-w\rangle| \, dw\, d\varepsilon \right\}$$

is compact in $L^1(\Sigma_R)$, and the integrand above is nonnegative, one obtains, after interchanging the order of integration and passing to a subsequence if necessary,

$$\iint_{R^3\times S^2_+} Y_f^\pm f_n(t, x\pm\varepsilon, w)\langle \varepsilon, v-w\rangle \, d\varepsilon\, dw \xrightarrow[n\to\infty]{} g^\pm \quad \text{a.e. in } (t, x, v)\in \Sigma_R \tag{3.29}$$

for some measurable functions $g^\pm$. On the other hand, using the same argument as for the sequences in (3.21) gives

$$\iint_{R^3\times S^2_+} Y_f^\pm (P_\varepsilon^\pm f_n)\langle \varepsilon, v-w\rangle \, d\varepsilon\, dw \xrightarrow[n\to\infty]{} L^\pm(f) \qquad \text{weakly in } L^1(\Sigma_R) \tag{3.30}$$

Now, (3.29) in combination with (3.30) implies (3.28).

Next, I claim that

$$\iint_{R^3\times S^2_+} |Y_n^\pm X_n^\pm W_n - Y_f^\pm|\, (P_\varepsilon^\pm f_n)\, \langle \varepsilon, v-w\rangle \, d\varepsilon\, dw \xrightarrow[n\to\infty]{} 0 \qquad \text{in } L^1(\Sigma_R) \tag{3.31}$$

By the convergence indicated in (2.16) and the definitions of $X_n^\pm$ and W_n one has, for a subsequence if necessary,

$$|Y_n^\pm X_n^\pm W_n - Y_f^\pm| \xrightarrow[n\to\infty]{} 0 \qquad \text{a.e. in } t, x, v, w, \varepsilon$$

As before [see (3.21)], the sequence $\{(P_\varepsilon^\pm f_n)\langle \varepsilon, v-w\rangle\}$ is weakly compact in $L^1(N)$; thus, the convergence in (3.31) follows from (i) of Lemma 3.1. Now combination of (3.28) and (3.31) completes the proof of (3.18). ∎

Proposition 3.4. For all $R>0$ and all $0 \leqslant \varphi \in L^\infty(\Sigma)$, one has

$$\int_{R^3} E_n^\pm(f_n)\varphi\, dv \xrightarrow[n\to\infty]{} \int_{R^3} E^\pm(f)\varphi\, dv \qquad \text{in measure on } (0,T)\times B_R \tag{3.32}$$

In addition,

$$E^\pm(f)(t,x,\cdot)\in L^1(R_v^3) \qquad \text{a.e. in } (t,x)\in(0,T)\times R^3$$

and

$$\frac{1}{1+f}E^-(f)\in L^\infty((0,T);L^1(R^3\times B_R))$$

Proof. I first prove (3.32) for $E_n^-(f_n)$. Observe that (3.17) also implies that

$$\int_{R^3} f_n\psi_n\, dv \xrightarrow[n\to\infty]{} \int_{R^3} f\psi\, dv \qquad \text{in } L^1((0,T)\times R^3) \tag{3.33}$$

for $\psi_n \xrightarrow[n\to\infty]{} \psi$ a.e. in $(t,x,v)\in\Sigma$, and

$$\sup_{n\geqslant 1}\left\|\frac{\psi_n}{1+|v|}\right\|_{L^\infty(\Sigma)}<\infty$$

Indeed,

$$\int_{R^3} f_n\psi_n\, dv=\int_{R^3} f_n(\psi_n-\psi)\, dv+\int_{R^3} f_n\psi\, dv \tag{3.34}$$

To show that the first term in (3.34) converges to 0 in $L^1((0,T)\times R^3)$, use the same argument that was given in the proof of (3.31) [note that the sequence $\{(1+|v|)f_n\}$ is weakly compact]. Now, (3.17) completes the proof of (3.33).

Next, because of (3.17)–(3.18), the sequence

$$\left\{\frac{1}{1+L_0(f_n)}L_n^+(f_n)\right\}$$

with

$$L_0(f_n)=\iint_{R^3\times S^2}(1+|w|)f_n(t,x+a\varepsilon,w)\, d\varepsilon\, dw$$

converges to

$$\frac{1}{1+L_0(f)} L^+(f) \qquad \text{pointwise a.e. in } (t, x, v) \in \Sigma$$

Thus, for any $\varphi \in L^\infty(\Sigma)$, (3.33) with

$$\psi_n = \frac{1}{1+L_0(f_n)} L_n^+(f_n)\varphi$$

implies that

$$\frac{1}{1+L_0(f_n)} \int_{R^3} E_n^-(f_n)\varphi\, dv$$

$$\xrightarrow[n\to\infty]{} \frac{1}{1+L_0(f)} \int_{R^3} E^-(f)\varphi\, dv \qquad \text{in } L^1((0, T)\times R^3) \quad (3.35)$$

This completes the proof of (3.32) for $E_n^-(f_n)$. Observe that so far use has ot been made of the nonnegativity of φ.

By the change of the variables $(v, w) \rightleftharpoons (v', w')$ and $\varepsilon' = -\varepsilon$, one has

$$\int_{R^3} \varphi E_n^+(f_n)\, dv = \int_{R^3} f_n \left(\iint_{R^3\times S_+^2} \varphi' Y_n^+ X_n^+ (P_\varepsilon^+ f_n)\, W_n\langle \varepsilon, v-w\rangle\, d\varepsilon\, dw \right) dv \tag{3.36}$$

a.e. in $(t, x, v) \in \Sigma$. Here, $\varphi' = \varphi(t, x, v')$ and $v' = v - \varepsilon\langle \varepsilon, v-w\rangle$. Thus, the proof of (3.32) for $E_n^+(f_n)$ will be reduced to the case of (3.32) for $E_n^-(f_n)$ if

$$\iint_{R^3\times S_+^2} \varphi' Y_n^+ X_n^+ (P_\varepsilon^+ f_n)\, W_n\langle \varepsilon, v-w\rangle\, d\varepsilon\, dw$$

$$\xrightarrow[n\to\infty]{} \iint_{R^3\times S_+^2} \varphi' Y_f^+ (P_\varepsilon^+ f)\langle \varepsilon, v-w\rangle\, d\varepsilon\, dw \tag{3.37}$$

in $L^1(\Sigma_R)$. But convergence in (3.37) follows easily by using arguments very similar to those in the proof of (3.18). In particular, a crucial step is the pointwise convergence of the sequence

$$\left\{ \iint_{R^3\times S_+^2} \varphi' Y_f^+ (P_\varepsilon^+ f_n)\langle \varepsilon, v-w\rangle\, d\varepsilon\, dw \right\} \tag{3.38}$$

which results from the nonnegativity of the integrand [see the proof of (3.29)].

Finally, convergence in measure in (3.32) implies that $E^{\pm}(f)(t, x, \cdot) \in L^1(R_v^3)$ a.e. in $(t, x) \in (0, T) \times R^3$. The last inclusion follows from the non-negativity of $f \in L^1(\Sigma)$. ∎

The next two propositions show that f, the weak limit of $\{f_n\}$, satisfies (2.12) and (2.14) with the equality signs replaced by inequalities.

Proposition 3.5. For all $0 \leqslant s \leqslant t \leqslant T$, f satisfies the following inequality:

$$f^{\#}(t, x, v) - f^{\#}(s, x, v) \exp\{-[F^{\#}(t) - F^{\#}(s)]\}$$
$$\geqslant \int_s^t E^+(f)^{\#}\,(\tau, x, v) \exp\{-[F^{\#}(t) - F^{\#}(\tau)]\}\,d\tau \qquad (3.39)$$

a.e. in $(x, v) \in R^3 \times R^3$, where $F^{\#}(t, x, v) = \int_0^t L^+(f)^{\#}\,(\tau, x, v)\,d\tau$.

Proof. Recall that f_n satisfies the exponential multiplier form (2.14) with $E(f)$ replaced by $E_n(f_n)$. By (3.18), the sequence $F_n^{\#}(t, x, v) = \int_0^t L_n^+(f_n)^{\#}\,(\tau, x, v)\,d\tau$ converges to $F^{\#}(t, x, v) = \int_0^t L^+(f)^{\#}\,(\tau, x, v)\,d\tau$ in $C([0, T]; L^1_{\mathrm{loc}}(R^3 \times R^3))$. In addition, the sequence $\exp\{-[F_n^{\#}(t) - F_n^{\#}(s)]\}$ is uniformly bounded by one and converges to $\exp\{-[F^{\#}(t) - F^{\#}(s)]\}$ in $L^1_{\mathrm{loc}}(R^3 \times R^3)$, uniformly in $0 \leqslant s \leqslant t \leqslant T$. Therefore, in order to prove (3.39), it is enough to show that for all $0 \leqslant \varphi \in L^\infty((0, T) \times \Sigma)$

$$\int_0^T \int_0^T \iint_{R^3 \times R^3} \varphi \left(\int_s^t E^+(f)^{\#}\,(\tau, x, v) \right.$$
$$\left. \times \exp\{-[F^{\#}(t) - F^{\#}(\tau)]\}\,d\tau \right) dv\,dx\,ds\,dt$$
$$\leqslant \liminf_{n \to \infty} \int_0^T \int_0^T \iint_{R^3 \times R^3} \varphi \left(\int_s^t E_n^+(f_n)^{\#}\,(\tau, x, v) \right.$$
$$\left. \times \exp\{-[F_n^{\#}(t) - F_n^{\#}(\tau)]\}\,d\tau \right) dv\,dx\,ds\,dt \qquad (3.40)$$

Note that the right-hand side of (3.40) is bounded by $2T \|f\|_{L^1(\Sigma)}$.

Next observe that

$$\iint_{R^3 \times R^3} \varphi \left(\int_s^t E_n^+(f_n)^{\#}\,(\tau, x, v) \exp\{-[F_n^{\#}(t) - F_n^{\#}(\tau)]\}\,d\tau \right) dv\,dx$$
$$= \int_s^t \iint_{R^3 \times R^3} \varphi(t, s, x - \tau v, v)\, E_n^+(f_n)\, G_n(t, \tau, x, v)\,dv\,dx\,d\tau$$

where

$$G_n(t, \tau, x, v) = \exp\{-[F_n(t, x+(t-\tau)v, v) - F_n(\tau, x, v)]\}, \qquad 0 \leqslant G_n \leqslant 1$$

and $G_n \xrightarrow[n \to \infty]{} G$ pointwise a.e., with

$$G(t, \tau, x, v) = \exp\{-[F(t, x+(t-\tau)v, v) - F(\tau, x, v)]\}$$

Now consider the sequence $f_n^R = \min\{f_n, R\}$ for $0 < R < \infty$. Since $f_n^R \leqslant f_n$, the sequence $\{f_n^R\}$ is weakly compact in $L^1(\Sigma)$. Because $0 \leqslant z - \min\{z, R\} \leqslant z\chi_{\{z \geqslant R\}}$ and $\{f_n\}$ is weakly compact, one also has

$$\sup_{n \geqslant 1} \sup_{t \in [0, T]} \|f_n - f_n^R\|_{L^1(R^3 \times R^3)} \xrightarrow[R \to \infty]{} 0 \tag{3.41}$$

Using the same argument as in (3.25) gives that $f^R \xrightarrow[R \to \infty]{} f$ in $L^\infty((0, T); L^1(R^3 \times R^3))$, where f^R is the weak limit of the sequence $\{f_n^R\}$.

Define E_n^R and $\tilde{E}_R^+(f)$ by

$$E_n^R = \iint_{R^3 \times S_+^2} Y_n^- X_R^- f_n^R(t, x, v') f_n(t, x - a\varepsilon, w') \langle \varepsilon, v - w \rangle W_R \, d\varepsilon \, dw \tag{3.42}$$

$$\tilde{E}_R^+(f) = \iint_{R^3 \times S_+^2} Y_f^- X_R^- f^R(t, x, v') f(t, x - a\varepsilon, w') \langle \varepsilon, v - w \rangle W_R \, d\varepsilon \, dw \tag{3.43}$$

I claim that for a fixed $R < \infty$, $0 \leqslant s \leqslant t \leqslant T$, and $0 \leqslant \varphi \in L^\infty((0, T) \times \Sigma)$ with $\tilde{\varphi} = \varphi(t, s, x - \tau v, v)$, one has

$$\lim_{n \to \infty} \int_s^t \iint_{R^3 \times R^3} \tilde{\varphi} E_n^R G_n \, dv \, dx \, d\tau = \int_s^t \iint_{R^3 \times R^3} \tilde{\varphi} \tilde{E}_R^+(f) G \, dv \, dx \, d\tau \tag{3.44}$$

Indeed, the change of variables $(v, w) \rightleftharpoons (v', w')$ and $\varepsilon' = -\varepsilon$ combined with similar arguments as in the proof of (3.18) leads to the following convergence for $0 \leqslant s \leqslant t \leqslant T$:

$$\begin{aligned} &\iint_{R^3 \times S_+^2} \varphi(t, s, x - \tau v', v') \, Y_n^- X_R^- f_n(\tau, x + a\varepsilon, w) \langle \varepsilon, v - w \rangle \\ &\qquad \times W_R G_n(t, \tau, x, v') \, d\varepsilon \, dw \\ &\xrightarrow[n \to \infty]{} \iint_{R^3 \times S_+^2} \varphi(t, s, x - \tau v', v') \, Y_f^- X_R^- f(\tau, x + a\varepsilon, w) \langle \varepsilon, v - w \rangle \\ &\qquad \times W_R G(t, \tau, x, v') \, d\varepsilon \, dw \end{aligned} \tag{3.45}$$

in $L^1((s,t)\times R^3\times R^3)$. Since $\lim_{n\to\infty} f_n^R = f^R$ weakly in $L^1(\Sigma)$, the limit in (3.44) follows from (ii) of Lemma 3.1. I am now ready to complete the proof of (3.40). For a fixed $R<\infty$ and $0\leqslant s\leqslant t\leqslant T$,

$$\begin{aligned}\liminf_{n\to\infty}\int_s^t\iint_{R^3\times R^3}&\tilde\varphi E_n^+(f_n)\,G_n\,dv\,dx\,d\tau\\ &\geqslant \lim_{n\to\infty}\int_s^t\iint_{R^3\times R^3}\tilde\varphi E_n^R G_n\,dv\,dx\,d\tau\\ &=\int_s^t\iint_{R^3\times R^3}\tilde\varphi \tilde E_R^+(f)G\,dv\,dx\,d\tau\end{aligned}\tag{3.46}$$

Since $f^R\uparrow f$ and $\tilde E_R^+(f)\uparrow E^+(f)$ pointwise a.e. as $R\uparrow\infty$, the monotone convergence theorem, integration with respect to t and s, and Fatou's lemma complete the proof of (3.40) and consequently the proof of Proposition 3.5. ∎

Corollary 3.6. For each $T>0$ and a.e. in $(x,v)\in R^3\times R^3$,

$$E^+(f)^{\#}\in L^1(0,T)\tag{3.47}$$

Proof. For each $T>0$ and a.e. in $(x,v)\in R^3\times R^3$,

$$\int_0^T E^+(f)^{\#}\,ds\leqslant \exp[F^{\#}(T)]\int_0^T E^+(f)^{\#}\exp\{-[F^{\#}(T)-F^{\#}(s)]\}\,ds$$

But $F^{\#}(s)\leqslant F^{\#}(t)$ a.e. in $(x,v)\in R^3\times R^3$ and for $0\leqslant s\leqslant t\leqslant T$. Since $F^{\#}(T)\in L^1(\Sigma_R)$ for any $R>0$, inequality (3.39) completes the proof. ∎

Proposition 3.7. For all $0\leqslant s\leqslant t\leqslant T$ and a.e. in $(x,v)\in R^3\times R^3$, f satisfies the inequality

$$f^{\#}(t,x,v)-f^{\#}(s,x,v)\leqslant\int_s^t E(f)^{\#}\,(\tau,x,v)\,d\tau\tag{3.48}$$

Proof. Observe that, for all $0\leqslant s\leqslant t\leqslant T$ and a.e. in $(x,v)\in R^3\times R^3$, f_n^δ satisfies

$$f_n^{\delta\#}(t,x,v)-f_n^{\delta\#}(s,x,v)=\int_s^t\left(\left[\frac{E_n^+(f_n)}{1+\delta f_n}\right]^{\#}-\left[\frac{f_n}{1+\delta f_n}\right]^{\#}L_n^+(f_n)^{\#}\right)d\tau\tag{3.49}$$

Using (3.18), the weak convergence of f_n^δ to f^δ, the weak convergence of $E_n^+(f_n)/(1+\delta f_n)$ and $h_n^\delta=f_n/(1+\delta f_n)$ to some E_δ^+ and h^δ, respectively, yields for $0\leqslant s\leqslant t\leqslant T$ and a.e. in $(x,v)\in R^3\times R^3$

$$f^{\delta\#}(t,x,v)-f^{\delta\#}(s,x,v)=\int_s^t[E_\delta^{+\#}-h^{\delta\#}L^+(f)^{\#}]\,d\tau\tag{3.50}$$

By (3.24)–(3.25), $f^{\delta\#}$ converges to $f^{\#}$ in $L^1(R^3\times R^3)$ uniformly in $t\in[0,T]$. Now, since

$$0\leqslant z-\frac{z}{1+\delta z}\leqslant \delta zR+z\chi_{\{z\geqslant R\}}$$

and $\{f_n\}$ is weakly compact, one also has

$$\sup_{t\in[0,T]}\|f-h^\delta\|_{L^1(R^3\times R^3)}\leqslant \sup_{t\in[0,T]}\liminf_{n\to\infty}\|f_n-h_n^\delta\|_{L^1(R^3\times R^3)}\xrightarrow[\delta\to 0^+]{}0 \tag{3.51}$$

Furthermore, $h^\delta\uparrow f$ as $\delta\downarrow 0^+$, and (3.48) follows from (3.50) by letting $\delta\to 0^+$ and using the monotone convergence theorem, if it can be shown that

$$E_\delta^+\leqslant E^+(f)\qquad \text{a.e. in } (t,x,v)\in(0,T)\times R^3\times R^3 \tag{3.52}$$

The proof of (3.52) follows from (3.32). Indeed, after passing to a subsequence if necessary, one has, for $0\leqslant\varphi\in L^\infty(\Sigma)$,

$$\frac{\int_{R^3}E_n^+(f_n)\varphi\,dv}{1+L_0(f_n)}\xrightarrow[n\to\infty]{}\frac{\int_{R^3}E^+(f)\varphi\,dv}{1+L_0(f)}\qquad\text{a.e. in } (t,x,v)\in\Sigma \tag{3.53}$$

where $L_0(f_n)$ and $L_0(f)$ are the same as in (3.35). Since

$$\frac{\int_{R^3}E_n^+(f_n)\varphi\,dv}{1+L_0(f_n)}\leqslant\|\varphi\|_{L^\infty(\Sigma)}\int_{R^3}f_n\,dv$$

the sequence

$$\left\{\frac{\int_{R^3}E_n^+(f_n)\varphi\,dv}{1+L_0(f_n)}\right\}$$

is weakly compact in $L^1((0,T)\times R^3)$. Therefore,

$$\begin{aligned}\iiint_\Sigma\frac{E_\delta^+\varphi}{1+L_0(f)}\,dv\,dx\,dt&\leqslant\lim_{n\to\infty}\frac{E_n^+(f_n)\varphi}{1+L_0(f_n)}\,dv\,dx\,dt\\&=\iiint_\Sigma\frac{E^+(f)\varphi}{1+L_0(f)}\,dv\,dx\,dt\end{aligned} \tag{3.54}$$

Since $0\leqslant\varphi\in L^\infty(\Sigma)$ was arbitrary,

$$\frac{E_\delta^+}{1+L_0(f)}\leqslant\frac{E^+(f)}{1+L_0(f)}\qquad\text{a.e. in } (t,x,v)\in\Sigma \tag{3.55}$$

Now (3.55) completes the proof of (3.52) and also the proof of Proposition 3.7. ∎

Note that by (3.47), $E^+(f)^\# \in L^1(0, T)$, a.e. in $(x, v) \in R^3 \times R^3$. Thus, (3.48) implies that $E^-(f)^\# \in L^1(0, T)$, a.e. in $(x, v) \in R^3 \times R^3$. The function f satisfying (3.39) can be called a supersolution of the exponential multiplier form (2.14), while f satisfying (3.48) can be called a subsolution of the mild form (2.12).

I am now ready to prove the main result of this section.

Proof of Theorem 3.2. The function $F^\#(t)$ defined in (2.14) is absolutely continuous for almost all x, v, and $dF^\#/dt = L^+(f)^\#$ a.e. in t. By Propositions 3.5 and 3.7, $f^\#$ is absolutely continuous in t for almost all x, v. Therefore, $f^\# \exp F^\#$ is absolutely continuous in t for almost all x, v. Proposition 3.5 implies that

$$\frac{d}{dt}(f^\# \exp F^\#) \geqslant E^+(f)^\# \exp F^\# \qquad \text{a.e. in } t, \text{ for almost all } x, v \tag{3.56}$$

Thus,

$$\frac{d}{dt} f^\# \geqslant E(f)^\# \qquad \text{a.e. in } t, \text{ for almost all } x, v \tag{3.57}$$

Finally, for almost all x, v,

$$f^\#(t) - f^\#(s) \geqslant \int_s^t E(f)^\# \, d\tau \qquad \text{for} \quad 0 \leqslant s \leqslant t \tag{3.58}$$

Now Proposition 3.7 combined with (3.58) implies that f is a mild solution of (2.2). ∎

I end this section by indicating a certain continuity property of $t \mapsto f(t) \in L^1(R^3 \times R^3)$. This property has been observed by DiPerna and Lions[1] in the case of the Boltzmann equation. From Eq. (3.49) it follows that for $0 \leqslant s \leqslant t \leqslant T$ and $\delta > 0$

$$\| f_n^{\delta\#}(t) - f_n^{\delta\#}(s) \|_{L^1(R^3 \times R^3)} \leqslant \int_s^t \left\| \frac{E_n^+(f_n)}{1 + \delta f_n} \right\|_{L^1(R^3 \times R^3)} d\tau \tag{3.59}$$

Using (3.59) together with (3.24) gives that for each $\gamma > 0$ there exists $\tau > 0$ such that for $|t - s| \leqslant \tau$ and uniformly in $n \geqslant 1$ one has

$$\| f_n^\#(t) - f_n^\#(s) \|_{L^1(R^3 \times R^3)} \leqslant \gamma \tag{3.60}$$

By passing to the limit $n \to \infty$ in (3.60) and observing that a norm is lower weakly semicontinuous, one can deduce that $f^\# \in C([0, T]; L^1(R^3 \times R^3))$. Finally, using (3.16) and the fact that $U(\cdot)\cdot$ is a jointly strongly continuous group in $L^1(R^3 \times R^3)$, one easily obtains $f \in C([0, T]; L^1(R^3 \times R^3))$. Here, $(U(-t) f)(t, x, v) = f^\#(t, x, v)$.

4. EXISTENCE AND PROPERTIES OF APPROXIMATE SOLUTIONS

In this section I consider for each $n \geqslant 1$ the truncated Enskog equation

$$\begin{gathered}\frac{\partial f_n}{\partial t} + v\frac{\partial f_n}{\partial x} = \mathscr{E}_n(t, f_n) \equiv \mathscr{E}_n^+(t, f_n) - \mathscr{E}_n^-(t, f_n) \\ f_n(0, x, v) = f_{0n}(x, v), \qquad 0 < t \leqslant T\end{gathered} \tag{4.1}$$

Here,

$$f_{0n} = \max\left\{\min\{f_0, n\}, \frac{\rho}{n}\exp(-x^2 - v^2)\right\}$$

where f_0 is nonnegative and satisfies (3.2). Here ρ is a small, positive constant determined later (see the proof of Theorem 4.4). Observe that $f_{0n} \to f_0$ in $L^1((0, T) \times R^3 \times R^3)$ as $n \to \infty$ and, for each $n \geqslant 1$, $f_{0n} \in L^\infty((0, T) \times R^3 \times R^3)$. Furthermore, $\mathscr{E}_n^\pm(t, f_n)$ are $E_n^\pm(f_n)$ [see (3.1)] with $Y_n^\pm$ replaced by

$$\begin{aligned}\mathscr{Y}_n^\pm(t, f_n) \equiv & \left[1 + \frac{1}{n}\int_{R^3} f_n(t, x, v)\, dv\right]^{-1} \\ & \times \left[1 + \frac{1}{n}\int_{R^3} f_n(t, x \pm a\varepsilon, v)\, dv\right]^{-1} \times Y_n^\pm\end{aligned} \tag{4.2}$$

The explicit dependence of E_n on t (through $Y_n^\pm$) has been suppressed in the notation up to this point. However, for the purpose of identifying (4.1) as a semilinear evolution equation, I have introduced the explicit t dependence in $\mathscr{E}_n(t, f_n)$ of (4.1).

Note that $\mathscr{Y}_n$ is *symmetric*, *bounded* (see Section 2), and satisfies (2.16). Thus, the convergence result (Theorem 3.2) also holds for the truncated Enskog equation (4.1). I prove that if $\|f_0\|_{L^1(R^3 \times R^3)}$ is small enough, then for each $n \geqslant 1$, (4.1) has a unique nonnegative solution satisfying conditions (i)–(iii) of Theorem 3.2.

I start with several definitions that set up (4.1) in the framework of a semilinear evolution equation. Consider the operator $Af \equiv -v \cdot \nabla_x f$ in $L^1(R^3 \times R^3)$. Then A generates a strongly continuous semigroup $U(t)$ in $L^1(R^3 \times R^3)$. For $M > 0$, recall the set D_M defined in Section 2,

$$D_M = \left\{f \in L^1(R^3 \times R^3): f \geqslant 0, \iint_{R^3 \times R^3} (1 + v^2) f\, dv\, dx \leqslant M\right\}$$

Note that D_M is closed in $L^1(R^3 \times R^3)$. Now we can rewrite (4.1) with

$F(t, f) = \mathscr{E}_n(t, f)$ in the form of a semilinear evolution equation in $L^1(R^3 \times R^3)$

$$\frac{d}{dt} f(t) + Af = F(t, f(t)), \qquad f(0) = f_0, \quad 0 < t \leqslant T \tag{4.3}$$

Note that in (4.3) the subscript n has been suppressed in f and f_0. One says that a continuous function f from $[0, T]$ into $D_M \subset L^1(R^3 \times R^3)$ is a weak solution of (4.3) if it satisfies

$$f(t) = U(t) f_0 + \int_0^t U(t-s) F(s, f(s)) \, ds \tag{4.4}$$

for $t \in [0, T]$. The integral in (4.4) is the Riemann integral in $L^1(R^3 \times R^3)$. In the literature a weak solution of (4.3), defined above, is often called a mild solution. Here, however, this name has been reserved for a different kind of solution defined in (2.12). Observe that a weak solution of (4.3) is always a mild solution. Indeed, it follows from the fact that, for each real t, $U(t)^{-1}$ exists and is equal to $U(-t)$, and $f^{\#}(t, x, v) = (U(-t) f)(t, x, v)$.

There are many theorems that guarantee the existence of weak solutions to semilinear evolution equations of the form given in (4.3) (see, for example, Chapter 8 of ref. 18, and in particular Theorem 2.1, p. 335 of ref. 18).

Theorem 4.1. With the notation as above and $f_0 \in D_M$ assume that:

(i) $U(t)$ is a strongly continuous semigroup in $L^1(R^3 \times R^3)$ generated by A, and $U(t)\colon D_M \to D_M$.

(ii) $F\colon [0, T] \times D_M \to D_M$ is continuous and there exists $K > 0$ such that

$$\|F(t, f) - F(t, g)\|_{L^1(R^3 \times R^3)} \leqslant K \, \|f - g\|_{L^1(R^3 \times R^3)}$$

for $0 \leqslant t \leqslant T$ and $f, g \in D_M$.

(iii) For $(t, f) \in [0, T] \times D_M$

$$\liminf_{h \to 0^+} \frac{1}{h} \operatorname{dist}(f + hF(t, f); D_M) = 0$$

where

$$\operatorname{dist}(f; D_M) = \inf_{g \in D_M} \|f - g\|_{L^1(R^3 \times R^3)}$$

which is the distance of f from D_M. Then there exists a unique weak solution f of (4.3) on $[0, T]$ for any $T>0$.

Note that assumption (iii), known as the Nagumo boundary condition, guarantees that a solution starting from $f_0 \in D_M$ stays in D_M.

Now it is easy to check that for each $n \geqslant 1$ and with $F(t, f) = \mathscr{E}_n(t, f)$, assumptions (i)–(iii) of Theorem 4.1 are satisfied. Indeed, it has already been indicated that $U(t)$ is a strongly continuous semigroup in $L^1(R^3 \times R^3)$. Since

$$\|(1+v^2)\, U(t) f\|_{L^1(R^3 \times R^3)} = \|(1+v^2) f\|_{L^1(R^3 \times R^3)}$$

one can deduce the second condition of (i). Next, using (2.15) and (4.2), one easily obtains (ii). Note that K in (ii) depends on n and $K \to \infty$ as $n \to \infty$. Finally, due to the form of $\mathscr{E}_n^{\pm}(t, f)$, one has for $t \geqslant 0$ and $f \in D_M$, $f + h\mathscr{E}_n(t, f) \geqslant 0$ for small enough $h > 0$; therefore, using (2.5) with $\psi = 1 + v^2$ yields (iii).

Consider the following norm, introduced by Arkeryd,[13] in the context of the Enskog equation:

$$\|f\|_E \equiv \iint_{R^3 \times R^3} [\operatorname*{ess\,sup}_{t \in [0, T]} |(U(-t) f)(t, x, v)|]\, dv\, dx \tag{4.5}$$

where f is measurable and a.e. finite on $(0. T) \times R^3 \times R^3$. Recall that $(U(-t) f)(t, x, v) = f(t, x + tv, v)$. This norm can be used to bound the integral in (i) of Theorems 3.2. Indeed, one has the following result.

Lemma 4.2. If f_n is a weak solution of (4.1), then

$$\int_0^T I_n^{\pm}(s)\, ds \leqslant \frac{C^{\pm}}{a^2} \|f_n\|_E^2 \tag{4.6}$$

where $C^{\pm} = \sup_{n \geqslant 1} \|\mathscr{Y}_n^{\pm}(s, f_n)\|_{L^\infty(W)}$.

Proof. First, by a simple change of variables $x \to x + sv$, one obtains

$$\begin{aligned} \int_0^T I_n^{\pm}(s)\, ds \leqslant C^{\pm} \iint_{R^3 \times R^3} & [\sup_{0 \leqslant s \leqslant T} f_n(s, x + sv, v)] \\ & \times \left[\iiint_{R^3 \times [0, T] \times S_+^2} f_n^{\#}(s, x + s(v - w) \pm a\varepsilon, w) \right. \\ & \left. \times \langle \varepsilon, v - w \rangle\, d\varepsilon\, ds\, dw \right] dv\, dx \end{aligned} \tag{4.7}$$

Next, following Cercignani[20] (see also Arkeryd[13]), one has that for fixed x, v, and w the Jacobian of the transformation $(\varepsilon, s) \xrightarrow{V} y = x + s(v - w) \pm a\varepsilon$ is $\pm (a^2\langle \varepsilon, v - w\rangle)^{-1}$. Therefore,

$$\int_0^T I_n^{\pm}(s)\, ds$$

$$\leqslant \frac{C^{\pm}}{a^2} \|f_n\|_E \sup_{(x,v)\in R^3\times R^3} \left[\iint_{R^3 \times V^{-1}([0,T]\times S_+^2)} f_n(s(y), y + s(y)w, w)\, dy\, dw\right]$$

$$\leqslant \frac{C^{\pm}}{a^2} \|f_n\|_E^2 \tag{4.8}$$

which completes the proof. ∎

By Lemma 4.2, in order to show that condition (i) of Theorem 3.2 is satisfied, it is enough to prove that $\sup_{n\geqslant 1} \|f_n\|_E < \infty$. Note that Arkeryd[13] used the norm $\|\cdot\|_E$ to obtain an existence theorem for the Enskog equation in the case of $Y \equiv \text{const}$. Here, the weak compactness argument is a main tool of the proof, and therefore $\|\cdot\|_E$ is used only to obtain additional properties of the solutions of the truncated problem (4.1). Observe that the results of Arkeryd[13] for $Y \equiv \text{const}$ cannot be extended to the general form of Y considered here.

I proceed to two preliminary results on the sequence $\{f_n\}$ of approximate solutions of the truncated Enskog equation (4.1).

Proposition 4.3. For each $n \geqslant 1$, $f_n \in L^\infty((0, T) \times R^3 \times R^3)$.

Proof. First, note that

$$\mathscr{E}_n^+(t, f_n) \leqslant C_1 \left[1 + \frac{1}{n}\int_{R^3} f_n(t, x, v)\, dv\right]^{-1}$$

$$\times \iint_{R^3\times S_+^2} f_n(t, x, v') f_n(t, x - a\varepsilon, w') \langle \varepsilon, v - w\rangle\, W_n\, d\varepsilon\, dw$$

$$\equiv C_1 \left[1 + \frac{1}{n}\int_{R^3} f_n(t, x, v)\, dv\right]^{-1} \tilde{Q}_n^+(f_n) \tag{4.9}$$

where

$$C_1 = \sup_{n\geqslant 1} \sup_{f\in D_M} \|Y_n^-(t, f)\|_{L^\infty(W)} < \infty$$

with $W = [0, T] \times R^3 \times R^3 \times R^3 \times S_+^2$. To estimate the right-hand side of (4.9), one needs the following integral representation of $\tilde{Q}_n^+(f)$, originally obtained by Carleman (p. 32 of ref. 19) for the case of the Boltzmann collision operator ($a=0$), but easily generalized to the present case:

$$\tilde{Q}_n^+(f)(t, x, v) = \int_{R^3} f(t, x, v') \int_{P_{v,v'}} |w' - v'|^{-1} \left(1 + \frac{1}{n} \cos\theta\right)^{-1} \times \chi_n f(t, x - a(v - v')(|v - v'|)^{-1}, w')\, dE'\, dv' \qquad (4.10)$$

where $P_{v,\bar{v}} = \{z \in R^3: \langle \bar{v} - v, z - v\rangle = 0\}$, dP' denotes Lebesgue measure on $P_{v,v'}$, and $\chi_n = 1$ if $v'^2 + w'^2 \leqslant n$ and $\chi_n = 0$ otherwise. Now, combination of (4.2), (4.4), and (4.10), together with the Gronwall lemma, gives the L^∞-bound on f_n. This bound depends on n and becomes unbounded as $n \to \infty$. ▌

It is instructive to consider the corresponding truncated problem in the case of the Boltzmann equation. Corresponding to (4.1), the truncated Boltzmann problem is

$$\frac{\partial f_n}{\partial t} + v\frac{\partial f_n}{\partial x} = Q_n(f_n) \equiv Q_n^+(f_n) - Q_n^-(f_n), \qquad f_n(0, x, v) = f_{0n}(x, v), \quad 0 < t \leqslant T$$

with

$$Q_n^+(f_n) = \left[1 + \frac{1}{n}\int_{R^3} f_n(t, x, v)\, dv\right]^{-1} \times \iint_{R^3 \times S_+^2} f_n(t, x, v') f_n(t, x, w') B_n\, d\varepsilon\, dw$$

$$Q_n^-(f_n) = \left[1 + \frac{1}{n}\int_{R^3} f_n(t, x, v)\, dv\right]^{-1} \times \iint_{R^3 \times S_+^2} f_n(t, x, v) f_n(t, x, w) B_n\, d\varepsilon\, dw$$

and

$$B_n = \inf\{B(\theta, |v - w|), n\} \times \left[\cos^2(\theta)\left(\frac{1}{n} + \cos^2\theta\right)^{-1}\right] \times \inf\{1, |v - w|^{1/n}\} \times \chi_n$$

where $\chi_n = 1$ if $v^2 + w^2 \leqslant n$ and $\chi_n = 0$ otherwise. Here $B(\theta, |v - w|)$ is the scattering kernel with the usual angular cutoff. For inverse power potentials, $\mathscr{F}(r)=r^{-s}$, $B(\theta, |v-w|)=b(\theta)\,|v-w|^{(s-4)/s}$ with $s>2$. For the hard-sphere model, $B(\theta, |v-w|)=|v-w|\cos\theta=\langle v-w, \varepsilon\rangle$. The existence theorems for the Boltzmann equation provided in this paper, as well as in the original work of DiPerna and Lions,[1] cover all soft and hard potentials; in fact, they hold for all $B(\theta, |v-w|)=b(\theta)\,|v-w|^{\lambda}$ with $-3<\lambda<2$ and $\int_{S^2_+} b(\theta)\,d\varepsilon<\infty$. Note that a special form of $B_n(\theta, v-w)$ given above is needed to show L^{∞}-estimation of the solutions of the truncated problem. Indeed, with the notation as in (4.10), one has

$$\iint_{R^3\times S^2_+} f_n(t, x, v')\, f_n(t, x, w')\, B_n(\theta, |v-w|)\, d\varepsilon\, dw$$

$$=\int_{R^3} f_n(t, x, v') \int_{E_{v,v'}} f_n(t, x, w') \times B_n(\theta, |v'-w'|)\, |w'-v'|^{-2} \cos^{-2}\theta\, dE'\, dv'$$

Now, it follows that B_n removes singularities in $\cos^{-2}\theta$ and in $|w'-v'|^{-2}$, thus enabling one to obtain the relevant L^{∞}-estimation for f_n.

Theorem 4.4. Suppose that either

$$\|f_0\|_{L^1(R^3\times R^3)}<\frac{a^2}{4C^-}$$

or, uniformly in t, the sequence $\{f_n(t)\}$ is uniformly integrable. Then for any $a>0$

$$\sup_{n\geqslant 1} \|f_n\|_E \leqslant C_2(f_0, a, T, C^-)<\infty \tag{4.11}$$

Proof. For fixed $0\leqslant\tau<T$ and $R>0$ define

$$f_n^{b\tau}(t, x, v)=\begin{cases}\min\{f_n(\tau, x-tv, v), R\}, & |v|\leqslant R\\ 0, & \text{otherwise}\end{cases} \tag{4.12}$$

Consider the decomposition of f_n into $f_n=f_n^{u\tau}+f_n^{b\tau}$, where $f_n^{u\tau}=f_n-f_n^{b\tau}$. In the case $\tau=0$ this decomposition has been introduced by Arkeryd.[13] In addition, define $f_n^{0\tau}\equiv f_n(\tau, x, v)-f_n^{b\tau}(t, x+tv, v)$ and $f^{00}\equiv f_n^{00}$.

The idea of the proof is to show that there is a finite partition $0=t_0<t_1<t_2<\cdots<t_N=T$ with the property that for any $1\leqslant i\leqslant N$, the norm $\|f_n\|_E$ restricted to $[t_{i-1}, t_i]$ is bounded uniformly in $n\geqslant 1$. For this purpose define

$${}^{i}\|g\|_E\equiv\iint_{R^3\times R^3} \Big[\operatorname*{ess\,sup}_{t\in[t_{i-1}, t_i]} |(U(-t)\,g)(t, x, v)|\Big]\, dv\, dx \tag{4.13}$$

Next, since for any $i \geqslant 0$ and $n \geqslant 1$,

$$^{i+1}\| f_n^{bt_i} \|_E \leqslant \| f_0 \|_{L^1(R^3 \times R^3)}$$

it will be enough to estimate $^{i+1}\| f_n^{ut_i} \|_E$. Let us start the process of finding the partition $\{t_i\}$. Using the nonnegativity of f_n, (4.4), and changing variables of integration, one has for any $0 < t_1 \leqslant T$

$$^{1}\| f_n^{u0} \|_E \leqslant \| f^{00} \|_{L^1(R^3 \times R^3)} + C^- \int_0^{t_1} \iiiint_{R^3 \times R^3 \times R^3 \times S_+^2} f_n(s, x, v) \\ \times f_n(s, x + a\varepsilon, w) \langle \varepsilon, v - w \rangle \chi_n \, d\varepsilon \, dw \, dv \, dx \, ds \tag{4.14}$$

where χ_n, defined in Section 3 [see Eq. (3.1)], makes the right-hand side of (4.14) finite. Recall that for each $n \geqslant 1$, $f_n \in L^\infty(\Sigma)$. Using the decomposition $f_n = f_n^{u0} + f_n^{b0}$ in the right-hand side of (4.14), together with the change of variables $v \rightleftharpoons w$, $\varepsilon \to -\varepsilon$, and $x - a\varepsilon \to x$, one obtains

$$\begin{aligned} ^{1}\| f_n^{u0} \|_E &\leqslant \| f^{00} \|_{L^1(R^3 \times R^3)} \\ &\quad + C^- \int_0^{t_1} \iiiint_{R^3 \times R^3 \times R^3 \times S_+^2} \{ f_n^{b0} f_{n*}^{b0} + 2 \, | f_n^{u0} | \, f_{n*}^{b0} + | f_n^{u0} | \cdot | f_{n*}^{u0} | \} \\ &\quad \times \langle \varepsilon, v - w \rangle \chi_n \, d\varepsilon \, dw \, dv \, dx \, ds \\ &= \| f^{00} \|_{L^1(R^3 \times R^3)} + P_1^0 + P_2^0 + P_3^0 \end{aligned} \tag{4.15}$$

where $*$ denotes that the function is evaluated at the point $(s, x + a\varepsilon, w)$. For P_1^0 one has the bound

$$P_1^0 \leqslant \frac{32\pi^2 R^4 (1 + R) \, t_1}{3} C^- \, \| (1 + v^2) f_0 \|_{L^1(R^3 \times R^3)}$$

Next,

$$P_2^0 \leqslant \frac{128\pi^2 R^4 (1 + R) \, t_1}{3} C^- \, \| (1 + v^2) f_0 \|_{L^1(R^3 \times R^3)}$$

and using an argument very similar to the one used in (4.6)–(4.8), one obtains

$$P_3^0 \leqslant \frac{C^-}{a^2} \, ^{1}\| f_n^{u0} \|_E^2$$

Using the inequality $0 \leqslant z - \min\{z, R\} \leqslant z \chi_{\{z \geqslant R\}}$, one can deduce that $\| f^{00} \|_{L^1(R^3 \times R^3)}$ converges to zero as $R \to \infty$. Therefore, from (4.15) and the bounds for P_k^0, $k=1, 2$, it follows that there exists $t_1>0$ small enough for which $\sup_{n\geqslant 1}{}^1\|f_n^{u0}\|_E\leqslant B_1<\infty$, and B_1 depends only on a, $f_0\in D_M$, and C^-.

Now suppose that one has found t_i for which $\sup_{n\geqslant 1}{}^i\|f_n^{ut_{i-1}}\|_E\leqslant B_i<\infty$ and $t_i<T$. I claim that it is possible to find $t_i<t_{i+1}\leqslant T$ for which $\sup_{n\geqslant 1}{}^{i+1}\|f_n^{ut_i}\|_E\leqslant B_{i+1}<\infty$. Indeed, instead of (4.15), one has for any $t_{i+1}\in(t_i, T]$

$$\begin{aligned}
{}^{i+1}\|f_n^{ut_i}\|_E &\leqslant \|f_n^{0t_i}\|_{L^1(R^3\times R^3)} \\
&\quad+\int_{t_i}^{t_{i+1}}\iiiint_{R^3\times R^3\times R^3\times S_+^2}\{f_n^{bt_i}f_{n*}^{bt_i}+2\,|f_n^{ut_i}|\,f_{n*}^{bt_i}+|f_n^{ut_i}|\cdot|f_{n*}^{ut_i}|\}\\
&\quad\times\langle\varepsilon, v-w\rangle\,\chi_n\,d\varepsilon\,dw\,dv\,dx\,ds\\
&=\|f_n^{0t_i}\|_{L^1(R^3\times R^3)}+P_1^i+P_2^i+P_3^i
\end{aligned}\tag{4.16}$$

The bounds for P_k^i are the same as for P_k^0 with $k=1, 2$, except that t_1 is replaced by $(t_{i+1}-t_i)$. Similarly, one obtains

$$P_3^i\leqslant\frac{C^-}{a^2}\,{}^{i+1}\|f_n^{ut_i}\|_E^2$$

Now, in the case when $\{f_n(t)\}$ is uniformly integrable, the inequality $0\leqslant z-\min\{z, R\}\leqslant z\chi_{\{z\geqslant R\}}$ together with the fact that

$$\iint_{R^3\times R^3}(1+v^2)\,f_n\,dv\,dx=\iint_{R^3\times R^3}(1+v^2)\,f_0\,dv\,dx$$

implies that $\|f_n^{0t_i}\|_{L^1(R^3\times R^3)}$ converges to zero as $R\to\infty$, uniformly in $n\geqslant 1$ and $t_i\in[0, T]$. In the case when

$$\|f_0\|_{L^1(R^3\times R^3)}<\frac{a^2}{4C^-}$$

by choosing ρ in the definition of f_{0n} small enough [see (4.1)], one has

$$\|f_n^{0t_i}\|_{L^1(R^3\times R^3)}<\frac{a^2}{4C^-}\qquad\text{uniformly in } t_i\in[0, T]\text{ and } n\geqslant 1$$

Therefore, we can find $t_{i+1}>t_i$, with $t_{i+1}-t_i$ small enough, for which ${}^{i+1}\|f_n^{ut_i}\|_E\leqslant B_{i+1}<\infty$, where B_{i+1} depends only on a, $f_0\in D_M$, and C^-. This process can be continued until $t_N=T$. Finally, the inequality

$$\|f_n\|_E\leqslant\sum_{i=1}^N{}^i\|f_n\|_E\leqslant N\,\|f_0\|_{L^1(R^3\times R^3)}+\sum_{i=1}^N B_i$$

completes the proof. ▮

The main result of this section is as follows.

Theorem 4.5. If $\|f_0\|_{L^1(R^3 \times R^3)} < a^2/4C^-$, then the sequence f_n of solutions of the truncated Enskog equation (4.1) satisfies conditions (i)–(iii) of Theorem 3.2.

Proof. Theorem 4.4 and Proposition 4.3 imply that conditions (i) and (iii) of Theorem 3.2 are satisfied. The proof of (ii) of Theorem 3.2 is divided into several steps. First, note that Eq. (4.4) implies

$$\begin{aligned}
&\iint_{R^3 \times R^3} x^2 U(-t) f_n(t)\, dv\, dx \\
&\quad = C_0 + \iint_{R^3 \times R^3} \int_0^t x^2 U(-s)\, \mathscr{E}_n(s, f_n(s))\, ds\, dv\, dx \\
&\quad = C_0 + \int_0^t \iint_{R^3 \times R^3} (x - sv)^2\, \mathscr{E}_n(s, f_n(s))\, dv\, dx\, ds \qquad (4.17)
\end{aligned}$$

where $C_0 = \iint_{R^3 \times R^3} x^2 f_0\, dv\, dx < \infty$. Using (2.5) with $\psi = (x - sv)^2$ and (2.9), we obtain that

$$\sup_{n \geq 1} \sup_{t \in [0, T]} \iint_{R^3 \times R^3} (x - tv)^2 f_n\, dv\, dx \leqslant C_T < \infty$$

Next, by repeating the same argument as in Section 3, one can deduce that

$$\sup_{n \geq 1} \sup_{t \in [0, T]} \iint_{R^3 \times R^3} x^2 f_n\, dv\, dx \leqslant C_3(T) < \infty \qquad (4.18)$$

Furthermore,

$$\begin{aligned}
&a \int_0^T \iiiint_{R^3 \times R^3 \times R^3 \times R^3} t \langle \varepsilon, v - w \rangle^2\, W_n \mathscr{Y}_n^+(t, f_n) \\
&\qquad \times f_n(t, x, v)\, f_n(t, x + a\varepsilon, w)\, d\varepsilon\, dw\, dv\, dx\, dt \\
&\quad \leqslant C_4(T) \qquad (4.19)
\end{aligned}$$

Note that $C_4(T)$ does *not* depend on n.

In view of Theorem 4.1 and the bound (4.18), it has been shown so far that

$$\sup_{n \geq 1} \sup_{t \in [0, T]} \iint_{R^3 \times R^3} (1 + v^2 + x^2)\, f_n(t, x, v)\, dv\, dx \leqslant C_5(T) < \infty \qquad (4.20)$$

In order to complete the proof of condition (ii) of Theorem 3.2, one needs to have control of f_n from below. Indeed, by multiplying Eq. (4.4) in its mild form by $1+\log f_n(t)^{\#}$ and integrating over Σ, one obtains

$$\iint_{R^3\times R^3}\int_0^T \frac{d}{dt}(f_n \log f_n)^{\#}\, dt\, dv\, dx$$
$$= \iiint_{[0,T]\times R^3\times R^3} \{\mathscr{E}_n(t, f_n(t))[\log f_n(t)+1]\}^{\#}\, dv\, dx\, dt \tag{4.21}$$

Since, for each $n\geqslant 1$, $(f_n)^{\#}\in L^{\infty}(\Sigma)$ and $\|(1+v^2)(f_n)^{\#}\|_{L^1(\Sigma)}\leqslant M$, the integrand on the right-hand side of (4.21) is integrable. However, in order to obtain the equality

$$\int_0^T \frac{d}{dt}(f_n \log f_n)(t)^{\#}\, dt$$
$$= (f_n \log f_n)(T)^{\#} - (f_{0n}\log f_{0n})^{\#} \qquad \text{a.e. in } (x, v)\in R^3\times R^3 \tag{4.22}$$

one needs the absolute continuity of $(f_n \log f_n)(t)^{\#}$ a.e. in $(x, v)\in R^3\times R^3$. Of course, once one knows that Eq. (4.22) holds, one can combine (3.8), Lemma 4.2, and Theorem 4.4 to obtain (ii) of Theorem 3.2.

Now, the absolute continuity can be proven by showing, for example, that for $n\geqslant 1$, and a.e. in $(x, v)\in R^3\times R^3$, $f_n(t)^{\#}\geqslant c(n, x, v)>0$, uniformly in $t\in[0, T]$. Indeed, since $\log z$ is Lipschitz continuous for $z\geqslant\alpha>0$, the absolute continuity of $f_n(t)^{\#}$ implies the absolute continuity $(f_n\log f_n)(t)^{\#}$ a.e. in $(x, v)\in R^3\times R^3$. To find a lower bound of $f_n^{\#}$, I proceed along the line of arguments given in ref. 1.

I claim that for all $t\in[0, T]$

$$f_n(t, x, v)\geqslant \frac{\rho}{n}\exp(-C_n t - |x-tv|^2 - v^2)\equiv g_n \qquad \text{a.e. in } (x, v)\in R^3\times R^3 \tag{4.23}$$

Since $\mathscr{E}_n^-(t, f_n)\leqslant C_n f_n$ on Σ, for some constant C_n [the same constant appearing in (4.23)], one can see that

$$\frac{\partial f_n}{\partial t} + v\frac{\partial f_n}{\partial x} + C_n f_n \geqslant 0 \qquad \text{in } \mathscr{D}'((0, T)\times R^3\times R^3) \tag{4.24}$$

Similarly, for g_n,

$$\frac{\partial g_n}{\partial t} + v\frac{\partial g_n}{\partial x} + C_n g_n = 0 \qquad \text{a.e. in } \Sigma \tag{4.25}$$

Now, since $f_n(0, x, v) \geqslant g_n(0, x, v)$, inequality (4.23) follows from (4.24) and (4.25). This completes the proof of (ii) of Theorem 3.2, and consequently the proof of Theorem 4.5. ∎

For arbitrary f_0 satisfying condition (3.2), one has the following result.

Corollary 4.6. If $T > 0$ is small enough, then the sequence f_n of solutions of the truncated Enskog equation (4.1) satisfies conditions (i)–(iii) of Theorem 3.2.

Proof. Choose $T = t_1$, where t_1 is defined in the first part of the proof of Theorem 4.4. It now follows that (4.11) is satisfied with T replaced by t_1. Next, identical arguments as in the proof of Theorem 4.5 complete the proof of Corollary 4.6. ∎

Note that, since the weak compactness in L^1 implies the uniform integrability, the proofs of Theorems 4.4 and 4.5 show that, for the sequence f_n of solutions of (4.1) given in Theorem 4.1,

$$\sup_{n \geqslant 1} \|f_n\|_E < \infty \Leftrightarrow \sup_{\substack{t \in [0, T] \\ n \geqslant 1}} \iint_{R^3 \times R^3} |\log f_n(t, x, v)| \, f_n(t, x, v) \, dv \, dx < \infty$$

Proofs can now be given of the main results of this paper, Theorems 2.1–2.3.

Proof of Theorem 2.1. For f_0 satisfying (2.18), first solve the truncated Enskog equation (4.1). In the case (1), Theorem 4.5, and in the case (2), Corollary 4.6, show that conditions (i)–(iii) of Theorem 3.2 are satisfied. Finally, Theorem 3.2 completes the proof. ∎

Proof of Theorem 2.2. It is enough to show that conditions (i)–(iii) of Theorem 3.2 are satisfied. By Lemma 4.2, condition (i) will be satisfied if it is shown that $\sup_{n \geqslant 1} \|f_n\|_E < \infty$. To do this, notice that, with the notation as in the proof of Theorem 4.4,

$$\|f_n\|_E \leqslant {}^1\|f_n\|_E + {}^{[t_1, T]}\|f_n\|_E \tag{4.26}$$

where the first term is the norm $\|f_n\|_E$ restricted to $[0, t_1]$ and with t_1 the same as in the proof of Theorem 4.4. Recall that t_1 is such that $\sup_{n \geqslant 1} {}^1\|f_n\|_E < \infty$. The second term in (4.26) is the norm $\|f_n\|_E$ restricted to $[t_1, T]$. To estimate it, notice that

$${}^{[t_1, T]}\|f_n\|_E \leqslant \|f_n(t_1)\|_{L^1(R^3 \times R^3)} + \int_{t_1}^{T} \iint_{R^3 \times R^3} \mathscr{E}_n^+(s, f_n(s)) \, dv \, dx \, ds \tag{4.27}$$

The change of variables $(v, w) \rightleftharpoons (v', w')$, $\varepsilon' = -\varepsilon$, together with an application of the integral form of (2.10) [with the truncated collision operator as in (4.1)] imply that one can bound the second term on the right-hand side of inequality (4.27) by $C_6/\gamma t_1$, with C_6 independent of n. This completes the proof that (i) of Theorem 3.2 is satisfied. Now one proceeds in exactly the same way as in the proof of Theorem 4.5 to show that conditions (ii)–(iii) of Theorem 3.2 are satisfied. ∎

Proof of Theorem 2.3. As in the proof of Theorem 2.2, it is enough to show that (i) of Theorem 3.2 is satisfied. Now, using a very similar argument as in (3.10) of ref. 14 gives that $\sup_{n \geqslant 1} \int_0^T I_n^+(s)\, ds < \infty$. ∎

Note that the compactness of the support of Y assumed in Theorem 2.3 can be replaced by a weaker condition on Y, already considered in ref. 14:

$$\sup_{\tau, \sigma \geqslant 0} \tau Y(\tau, \sigma) < \infty \tag{4.28}$$

Finally, Theorem 2.3 holds for bounded spatial domains with periodic boundary conditions. Indeed, notice that for $x \in \Omega = R^3/Z^3$, one also has $f^{\#}(t, x, v) = f(t, x + tv, v)$, where $x + tv$ is understood modulo 1. In addition, the operator $Af \equiv -v \cdot \nabla_x f$ generates a strongly continuous group $(U(-t) f)(t, x, v) = f^{\#}(t, x, v)$ in $L^1(\Omega \times R^3)$. Observe that in this case the explicit x^2 term appearing in condition (ii) of Theorem 3.2 (and also in Sections 3 and 4) is superfluous.

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